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Abstract: A family of boundary value problems is considered in domains $\Omega(\varepsilon) = \Omega \setminus \overline{\omega_\varepsilon} \subset \mathbb{R}^n$, $n \geq 3$, with cavities ω_ε depending on a small parameter $\varepsilon \in (0, \varepsilon_0]$. An approximation $\mathcal{U}(\varepsilon, x)$, $x \in \Omega(\varepsilon)$, of the solution $u(\varepsilon, x)$, $x \in \Omega(\varepsilon)$, to the boundary value problem is obtained by an application of the methods of matched and compound asymptotic expansions. The asymptotic expansion is constructed with precise a priori estimates for solutions and remainders in Hölder spaces, i.e., pointwise estimates are established as well. The asymptotic solution $\mathcal{U}(\varepsilon, x)$ is used in order to derive the first term of the asymptotic expansion with respect to ε for the shape functional $\mathcal{J}(\Xi(\varepsilon)) = \mathbb{J}_\varepsilon(u) \cong \mathbb{J}_\varepsilon(\mathcal{U})$. In particular, we obtain the *topological derivative* $\mathcal{T}(x)$ of the shape functional $\mathcal{J}(\Xi)$ at a point $x \in \Omega$. Volume and surface functionals are considered in the paper.

Key-words: asymptotic analysis, matched asymptotic expansion, compound asymptotic expansion, shape optimization, shape functional, topological derivative

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Analyse asymptotique de fonctionnelles de formes

Résumé : On considère dans $\mathbb{R}^n$, $n \geq 3$, une famille de problèmes définis dans des domaines $\Omega(\varepsilon) = \Omega \setminus \overline{\omega_\varepsilon}$ avec des trous ω_ε dépendant d'un paramètre $\varepsilon \in (0, \varepsilon_0]$. On obtient une approximation $\mathcal{U}(\varepsilon, x)$, $x \in \Omega(\varepsilon)$, de la solution $u(\varepsilon, x)$, $x \in \Omega(\varepsilon)$, à ce problème en appliquant les méthodes asymptotiques combinées à la procédure de réarangement des résidus. On construit le développement asymptotique et on établit des estimations a priori pour les solutions et les restes dans des espaces de Hölder, c'est-à-dire des estimations ponctuelles. La fonction $\mathcal{U}(\varepsilon, x)$ est utilisée afin de dériver par rapport à ε le premier terme du développement asymptotique de la fonctionnelle $\mathcal{J}(\Omega(\varepsilon)) = J_\varepsilon(u)$. On obtient en particulier la dérivée topologique $\mathcal{T}(x)$ de la fonctionnelle $\mathcal{J}(\Omega)$ en un point $x \in \Omega$. On considère également dans cet article des fonctionnelles de volume et de surface.

Mots-clés : analyse asymptotic, développement asymptotique raccordé, développement asymptotique composé (ou multi-échelle), optimisation de forme, fonctionnelle de forme, dérivée topologique

1 Introduction

1.1 Asymptotic analysis in domains with singularly perturbed boundaries

For the first time, the asymptotics of solutions to boundary value problems in domains with small holes (*cavities* in the three dimensional case) were constructed in [14], [15] (see also the book [16]) in the case of second order scalar equations by an application of the method of matched asymptotic expansions.

A different approach, using the method of compound asymptotic expansions combined with the specific procedure of rearrangements of discrepancies is proposed in [35]. The method is applied to systems of equations, elliptic in the sense of Douglis–Nireneberg, in domains with local geometrical perturbations in the form of holes and cavities or caverns and bulges near isolated boundary points and conical points. Full asymptotic expansions of solutions are constructed and precise a priori estimates in Sobolev spaces for solutions and remainders are established using the special construction of *almost–inverses* of boundary value problem operators.

A description of asymptotic algorithms can also be found in the books [37], [38], [39], [43], where many examples are provided in addition to the general theory of perturbations of geometrical domains.

Let us briefly describe some applications of the approach introduced in [35] and applied to many different singularly perturbed problems. Nonlinear equations are considered in [34], [46], spectral problems in domains with small holes are investigated in [40], [63], [36], [47], [48], [20], the asymptotics of energy functionals for general boundary value problems are constructed in [33], for isotropic elasticity in [71], [72], and for anisotropic elasticity in [55]. We refer the reader to [54], [8], [7], for other applications in elasticity theory. On the other hand, the problems with artificial boundary conditions, which are close to the subject of the paper, are studied in [56], [57], [58]. Finally, the method of asymptotic analysis has been applied, which is somehow surprising, for construction of selfadjoint extensions of differential operators in [53], [50], [20], [55], (see also chapter 6 in [43]), and for the description and modelling of elastic bodies with small defects in [5], [60] and other papers.

The formal asymptotic analysis performed in the present paper uses similar methods and it is described in a similar framework as in references [50], [20], but for different purposes. However, the estimates of asymptotic remainders are derived in *weighted Hölder norms.*

1.2 Preliminaries

Let Ω, ω be two domains in $\mathbb{R}^n$ with compact closures $\overline{\Omega}, \overline{\omega}$ and smooth boundaries $\partial\Omega, \partial\omega$. We assume that both domains Ω and ω contain the origin, and we have the following inclusions $\mathcal{O} = (0, ..., 0) \in \omega \subset \mathbb{B}_1 \subset \mathbb{B}_2 \subset \Omega$, where $\mathbb{B}_R = \{x \in \mathbb{R}^n \mid |x| \leq R\}$ denotes a ball of radius R. It is easy to achieve these inclusions by shifting the origin and rescaling. We introduce the sets

$$\omega_\varepsilon = \{x \in \mathbb{R}^n \mid \xi := \varepsilon^{-1} x \in \omega\}, \quad \Omega(\varepsilon) = \Omega \setminus \overline{\omega}_\varepsilon\,, \tag{1}$$

where $\varepsilon \in (0, 1]$ is a small parameter. By Σ we denote a smooth surface of dimension $d \leq n$, in the case of $d = n$ we have $\Sigma = \mathbb{R}^n$, in the case of $d = n - 1$, Σ becomes a hypersurface. Ξ denotes the part of Σ included in Ω, i.e., $\Xi = \Sigma \cap \Omega$, and we always assume that $\mathcal{O} \in \Xi$.

We shall consider the set

$$\Xi(\varepsilon) = \Xi \cap \Omega(\varepsilon)$$

depending on the small parameter ε and we introduce the integral shape functional

$$\mathbb{J}_\varepsilon(u) = \int_{\Xi(\varepsilon)} \mathcal{F}(x, u(\varepsilon, x), \nabla_x u(\varepsilon, x)) ds_x\,, \tag{2}$$

which depends first of all explicitly on ε since the integral is defined on the set $\Xi(\varepsilon)$, and also depends implicitly on ε since the integrand is a function of the trace of a vector function $u = (u_1, ..., u_T)$ and of its gradient $\nabla_x u$ on $\Xi(\varepsilon)$. Here u is a solution to the boundary value problem

$$\mathcal{L}(\nabla_x)u(\varepsilon, x) = f(\varepsilon, x), \quad x \in \Omega(\varepsilon) \ , \tag{3}$$

$$\mathcal{B}^{\Omega}(x, \nabla_x)u(\varepsilon, x) = g(\varepsilon, x), \quad x \in \partial\Omega \ , \tag{4}$$

$$\mathcal{B}^{\omega}(\varepsilon^{-1}x, \nabla_x)u(\varepsilon, x) = g^{\omega}(\varepsilon, x), \quad x \in \partial\omega_{\varepsilon} \ . \tag{5}$$

The operator $\mathcal{L}(\nabla_x)$ is a $T \times T$–matrix of homogeneous second order differential operators with constant coefficients,

$$\mathcal{L}(\nabla_{\xi}) = \mathcal{L}(\varepsilon^{-1}\nabla_x) = \varepsilon^{-2}\mathcal{L}(\nabla_x) \ . \tag{6}$$

Operator (6) is assumed to be formally self adjoint. Boundary operators (4) and (5) are sufficiently general, however should be chosen in such a way that problem (3)-(5) remains self adjoint, i.e. the Green's formula stays valid

$$\begin{aligned}(\mathcal{L}u, v)_{\Omega(\varepsilon)} + (\mathcal{B}^{\Omega}u, \mathcal{T}^{\Omega}v)_{\partial\Omega} + (\mathcal{B}^{\omega}u, \mathcal{T}^{\omega}v)_{\partial\omega_{\varepsilon}} = \\ (u, \mathcal{L}v)_{\Omega(\varepsilon)} + (\mathcal{T}^{\Omega}u, \mathcal{B}^{\Omega}v)_{\partial\Omega} + (\mathcal{T}^{\omega}u, \mathcal{B}^{\omega}v)_{\partial\omega_{\varepsilon}} \ .\end{aligned} \tag{7}$$

for all functions $u, v \in C^{\infty}(\overline{\Omega(\varepsilon)})^T$. Here $(\cdot, \cdot)_D$ stands for the scalar product in $L_2(D)$. A fairly used method of construction of operators $\mathcal{B}^{\Omega}$ and $\mathcal{B}^{\omega}$ is described in [30]. The following Green's formula is supposed to be valid in $\Omega(\varepsilon)$

$$(\mathcal{L}u, v)_{\Omega(\varepsilon)} + (\mathcal{N}^{\Omega}u, v)_{\partial\Omega} + (\mathcal{N}^{\omega}u, v)_{\partial\omega_{\varepsilon}} = a(u, v; \Omega(\varepsilon)) \ , \tag{8}$$

where $\mathcal{N}^{\Omega}, \mathcal{N}^{\omega}$ are $T \times T$ matrices of the Neumann boundary operators on $\partial\Omega$ and $\partial\omega_{\varepsilon}$, respectively. The sesquilinear form $a(\cdot, \cdot; \Omega(\varepsilon))$ is symmetric and non negative, i.e.,

$$a(\lambda u, \mu v) = \lambda\overline{\mu}a(u, v), \quad a(u, v) = \overline{a(v, u)}, \quad a(u, u) \geq 0 \ . \tag{9}$$

As usual bar denotes complex conjugation. The rows $\mathcal{B}_j^{\Omega}, \mathcal{T}_j^{\omega}$ of the $T \times T$ matrices $\mathcal{B}^{\Omega}, \mathcal{T}^{\omega}$ are selected in one of the following ways, separately for each row with index $j = 1, ..., T$,

$$\mathcal{B}_j^{\Omega}u = (\mathcal{S}^{\Omega}\mathcal{N}^{\Omega}u)_j \ , \ \mathcal{T}_j^{\Omega}u = (\mathcal{S}^{\Omega}u)_j \ , \tag{10}$$

$$or \quad \mathcal{B}_j^{\Omega}u = (\mathcal{S}^{\Omega}u)_j \ , \quad \mathcal{T}_j^{\Omega}u = -(\mathcal{S}^{\Omega}\mathcal{N}^{\Omega}u)_j \ . \tag{11}$$

Here $\mathcal{S}^{\Omega} \in C^{\infty}(\partial\Omega)^{T\times T}$ is orthogonal matrix function on $\partial\Omega$. Similarly, the matrices $\mathcal{B}^{\omega}, \mathcal{T}^{\omega}$ are defined by replacing $\mathcal{S}^{\Omega}(x)$ by $\mathcal{S}^{\omega}(\xi)$ which depends on the *fast* variable $\xi \in \partial\omega$. From (8),(10)–(11) the *full* Green's formula (7) follows.

Let us point out that we establish general properties of the linear mapping associated to problem (3)-(5) and to this end we should consider general right hand sides f, g and g^{ω} for the problem. However, for evaluation of functional (2) we need only the particular form of the right hand sides. Actually, we assume that $g^{\omega} = 0$ and $f = f^{\Omega}$ is independent of ε and defined everywhere in Ω, in particular in the cavity ω_{ε}. In addition,

$$g(\varepsilon, x) = g^{\Omega}(x) + \widehat{g}(\varepsilon, x) \ , \tag{12}$$

where $g^{\Omega}(x)$ is a function independent of ε, and $\widehat{g}$ a small correction, which serves to ensure the fulfilment of compability conditions (39). In the particular cases of the explicit solvability of problem (3)-(5), i.e., if the unique solution exists, or with the condition $\mathrm{supp} f^{\Omega} \cap \omega_{\varepsilon} = \emptyset$ the function $\widehat{g} = 0$ is admissible. Condition (12) is physically acceptable, since e.g., the appearance of a cavity in the elastic body in the state of the static equilibrium must diminish the exterior loading by the decrease of the weight.

1.3 Shape and topology optimisation

Let us outline the usage of the topological derivatives for numerical solutions of shape optimisation problems. In the classical theory of shape optimisation [65] necessary optimality conditions are given at the boundary of an optimal domain. The optimality conditions are derived applying shape calculus and shape gradients describing the boundary variations. Shape calculus is used as well in order to improve the value of shape functionals during the process of optimisation. Unfortunately, there is no result which could be applied in order to establish the optimality conditions in the interior of an optimal domain and could be used to improve the topology of the domain. Such results require the analysis of *topology variations*. In [66] asymptotic analysis is used in order to derive the explicit form of the so–called *topological derivatives* of shape functionals in two dimensions. The topological variations are defined in the form of small disks in two dimensions and constructive formulae are established for scalar elliptic equations as well as for the two dimensional isotropic elasticity system. The method of proof was based on shape calculus, and asymptotic expansions of solutions to elliptic boundary value problems with respect to the radius of a small hole were used. Similar results for the three dimensional elasticity system are given in [69] for topological variations defined by cavities in the form of balls of small radii. The formulae obtained for the three dimensional elasticity system are still constructive and can be used for numerical computations, however the formulae are no longer explicit in contrary to the two dimensional case.

Similar results are used for numerical solutions of inverse problems for the identification of small holes or inclusions [17], [18], [68]. The homogenization method in topology optimisation is presented e.g., in [1], [4]. Theoretical and numerical results related to the topology optimisation problems in two dimensional elasticity are given in [9], [11], [19], [26], [29].

We refer to papers [33], [71], [72], [55], where the energy functionals for elasticity problems are evaluated, and also to papers [13] and [48], [50], [55] concerning mathematical modelling of imperfections (microfissures, cracks, inclusions, etc) in elastic materials. The relation of the cited papers with our investigations are commented in sections 6.5, 6.6, where, besides that, methods of defect modelling in solids are discussed.

The asymptotic solution $\mathcal{U}(\varepsilon, x)$ to (3)-(5) constructed in this paper is used in order to derive first terms of the asymptotic expansion with respect to ε for the shape functional $\mathcal{J}(\Xi(\varepsilon)) = \mathbb{J}_\varepsilon(u)$ in (2). In particular, we obtain the form of the *topological derivative* $\mathcal{T}(\mathcal{O})$ of the shape functional $\mathcal{J}(\Xi)$ at the point $\mathcal{O} \in \Omega$

$$\mathcal{T}(\mathcal{O}) = \lim_{\varepsilon \to 0+} \frac{\mathcal{J}(\Xi(\varepsilon)) - \mathcal{J}(\Xi)}{\mathcal{M}(\omega_\varepsilon)} \ ,$$

where $\mathcal{M}(\omega_\varepsilon)$ is a measure of the cavity ω_ε. The appropriate choice of the term $\mathcal{M}(\omega_\varepsilon)$ follows directly from the asymptotic expansion and it depends on the shape functional and on the boundary conditions prescribed on the cavity ω_ε. For example, if $\Xi = \Omega$ and $\Xi(\varepsilon) = \Omega(\varepsilon)$, then $\mathcal{M}(\omega_\varepsilon) = (\mathrm{diam}\omega_\varepsilon)^n$ can be used for evaluation of the topological derivative of the integral shape functional $\mathcal{J}(\Omega)$ defined in Ω and depending on solutions to elliptic boundary value problems with Neumann boundary conditions prescribed on the cavity surface. In the case of Dirichlet boundary conditions on $\partial\omega_\varepsilon$ the classical capacity of ω_ε can be used, rather than the volume of the cavity, to define $\mathcal{M}(\omega_\varepsilon)$. Surface shape functionals are also considered and the proper choice of $\mathcal{M}(\omega_\varepsilon)$ is clearly made for such functionals.

1.4 Structure of paper and description of results

The construction of the asymptotic expansions is performed in few steps. In the first step the objective is to establish the asymptotic solution to problem (3)-(5) using the arguments of

[38] and some other papers, complemented for our specific purposes. In the second step the integral functional (2) is evaluated at the asymptotic expansion of solution. The most suitable framework for such a procedure is the scale of Hölder spaces which are used to establish the pointwise estimates of asymptotic accuracy for the solutions as well as for the derivatives of the solutions with respect to the spatial variables. Taking into account the singular perturbation of the geometrical domains, the most adapted setting includes the appropriate weighted norms in function spaces where solutions live. Actually, the linear operators of limit boundary value problems defined in domains $\Omega \setminus \mathcal{O}$ and $\mathbb{R}^n \setminus \overline{\omega}$ become isomorphisms in weighted Hölder spaces, Theorems 2.2 and 2.3.

The linear operator $\mathcal{A}^\varepsilon$ of the singularly perturbed problem has the crucial property, for our purposes, of uniform boundedness in the appropriate operator norm. This, together with the uniform boundedness of $(\mathcal{A}^\varepsilon)^{-1}$ is shown in Theorem 2.4. As it was mentioned above, up to now the elliptic boundary value problems in singularly perturbed domains are investigated to the full extend (see, e.g., monographs [37], [16], [25], [38], [39]), however the uniform a priori estimates, to the best knowledge of the authors, are not established in the existing literature in such a generality which is required for the purposes of the paper. The estimates in weighted Sobolev spaces are sufficient for most of applications, so there was no need for more complicated setting and more involved estimates which have to be used in the paper. Let us recall that in Sobolev spaces and in more general weighted Sobolev spaces, by an application of the Hardy inequality, the boundary value problems admit variational formulation.

For limit problems, independent of small parameter ε, the general results of [31] (see also [43], section 3.6) are used to establish properties of boundary value problems in weighted Hölder classes based on the corresponding properties in Sobolev classes. Such developments can be found in three subsection of section 2, in addition we exploit the so-called *polynomial property* of problem (3)-(5). The property is common for most of the boundary value problems of mathematical physics with symmetric bilinear forms (see [45], [52] and sections 5-7 in [43]). In last subsection 2.4 the main auxiliary result for the norm of the inverse linear operator associated to the singularly perturbed boundary value problem is proved. The argument applies the procedure of *gluing* of inverses of limit problems operators combined with the crucial *trick* of the removal of kernel and co-kernel of the mapping $\mathcal{A}^\varepsilon$ by means of introduction of some nonlocal compact perturbation. It results in new boundary value problems (46)-(48). Such approach allows us to minimise the technicalities connected with the analysis of problem (3)-(5) which is not uniquely solvable, and arbitrariness of the solution selection in such a case.

The asymptotic analysis of (3)-(5) is performed itself in section 4. However, all necessary auxiliary results are collected in section 3, where we also introduce the special singular solutions to homogeneous limit problems which are used to construct the proposed approximation of $u(x, \varepsilon)$. The most important object in the analysis, the so-called *polarisation matrix* is introduced and some properties of the matrix are established. The polarisation matrices are present, either in explicit forms or implicitly in all asymptotic formulae obtained in the paper. The polarisation matrices are attributes of boundary surfaces $\partial\Omega$, $\partial\omega$ and differential operators $\mathcal{L}, \mathcal{B}^\Omega, \mathcal{B}^\omega$. Such matrices are in fact generalisations of classical objects in harmonic analysis like the capacity, the polarisation tensor and the tensor of virtual masses due to Polya, Schiffer and Szegö [62], [61]. In section 3 we recall only the known results necessary for our purposes, we refer the reader for details to papers [49], [20] in the case of general boundary value problems, and to [71], [55] in the case of elasticity system.

The specific matrix notation used for construction of asymptotic solutions, first introduced in [50], can be explained and justified as follows. First of all, the special solutions of first and second limit problems, used throughout the paper are collected in rows, see e.g. section 4.

The matching procedure applied for derivation of asymptotic approximation $\mathcal{U}$ results in two columns $a(\varepsilon)$, $b(\varepsilon)$ which contain the coefficients of linear combinations of special solutions. Formal description of the procedure and coefficients can be relatively complex, see e.g. [16], in our notation such procedure is restricted to elementary operations on matrices and columns. The notation is used in final formulae (102), (103) given by Lemma 4.1 for columns $a(\varepsilon)$ and $b(\varepsilon)$. The columns are constructed from the columns of coefficients of Taylor expansion written for the solution of the first limit problem in Ω in terms of the polarisation matrices m^{Ω}, m^{ω} and the diagonal normalising matrix $\mathcal{E} = \mathcal{E}(\varepsilon)$ defined in (98). For our specific purposes, the required *accuracy of asymptotic approximation* is discussed in section 4.4. The estimate of the norm for the difference $u - \mathcal{U}$ between the exact solution and its approximation is given in Theorem 4.1. We point out that the simplicity of the argument in the latter theorem is partially due to the matrix notation, which simplifies the evaluation of the discrepancies of the asymptotic solution $\mathcal{U}$ in problem (3)- (5). The principal role is played by the uniform estimates for solutions to problem (3)-(5) in section 2.

Asymptotic analysis of shape functional (2) is performed in section 5. The structure of asymptotic solution $\mathcal{U}(\varepsilon, x)$ readily indicates two different cases which result in different asymptotic expansions of shape functionals.

1° The boundary conditions prescribed on the interior surface $\partial\omega_\varepsilon$ (the boundary of cavity) are different from the Neumann conditions and the resulting perturbation of the solution v to the first limit problem in Ω is $O(1)$ with respect to ε. Since in such a case the gradient $\nabla_x u(\varepsilon, x)$ of the solution to singularly perturbed problem is of magnitude $(1 + |x|)^{-1}$, thus becomes equal to $O(\varepsilon^{-1})$ near to the cavity and grows boundlessly with $\varepsilon \to 0+$. For such a solution the method of linearisation devised in the paper is not applicable to functional (2). Therefore, we have to restrict ourselves to the particular case of shape functional

$$\mathbb{J}_\varepsilon(u) = \int_{\Xi(\varepsilon)} \mathcal{F}(x, u(\varepsilon, x)) ds_x \ , \tag{13}$$

such that the integrand $\mathcal{F}$ depends only on the function u.

2° The Neumann boundary conditions conditions are prescribed on the surface $\partial\omega_\varepsilon$. In such a case some terms of the derived asymptotic representation vanish, some of left columns and upper rows of the polarisation matrix m^{ω} reduce to the null columns and rows and the solution $u(\varepsilon, x)$ and its gradient $\nabla_x u(\varepsilon, x)$ are approximately equal to $v(x)$ and $\nabla_x v(x)$, respectively. Hence, the linearisation of shape functional (2) can be performed at the solution v of the first limit problem. Simultaneously, the lower order terms of twofold expansion should be taken into account, we mean here the solutions to the first and the second limit problems, so the procedure of evaluation of topological derivatives becomes quite technical. Thus, all details of the procedure including the evaluation of asymptotic discrepancies are relegated to the proof of Lemma 7.1 in Appendix.

The main results of the paper include Theorems 5.1 and 5.3 which present the leading terms of asymptotic expansions of functionals (13) and (2) and therefore the topological derivatives in cases **1**° and **2**°, respectively. The last section 6 concerns anisotropic theory of elasticity for a body with small cavity. The matrix form (without any tensor notation) of elasticity problem is fully consistent with the notation already used in the previous sections of the paper. In section 6 we provide the formulae for topological derivatives of volume functional (184) similar to the energy functional, and of surface functional (185) appearing by the Neuber-Novozhilov-Weighardt fracture criteria. In supplementary subsections of section 6 it is shown that our general results include as particular cases and generalises all former results [71], [72], [55], [69], [13].

2 Polynomial property and existence of solutions to boundary value problems

We start with a brief description of the general theory of elliptic boundary value problems with symmetric bilinear forms.

2.1 Polynomial property and solvability of the first limit problem

We say that a pair $\{\mathcal{L}, a\}$ has the *polynomial property* [45],[52], if the following assertion holds for an arbitrary domain $D \subset \mathbb{R}^n$,

$$(14) \qquad u \in C^\infty(\overline{D})^T, \quad \mathcal{L}(\nabla_x)u(x) = 0,\ x \in D, \quad a(u,u;D) = 0 \Leftrightarrow\ u \in P|_D\ .$$

Here P is a finite dimensional linear space of polynomials in $x = (x_1, ..., x_n)$ with coefficients in $\mathbb{C}^T$. In particular, for the elasticity system, P is the space of rigid motions (see section 6.1).

We denote

$$(15) \qquad P(\Omega) = \{p \in P\ :\ \mathcal{B}^\Omega(x, \nabla_x)p(x) = 0,\ x \in \partial\Omega\}\ ,$$

$$(16) \qquad P(\omega) = \{p \in P\ :\ \mathcal{B}^\omega(\varepsilon^{-1}x, \nabla_x)p(x) = 0,\ x \in \partial\omega_\varepsilon\}\ .$$

We introduce two limit problems in order to define the expansions of the solutions to problem (3)–(5) with respect to ε.
The first limit problem is obtained by filling the cavity ω_ε, so it is formulated in the unperturbed domain Ω,

$$(17) \qquad \mathcal{L}(\nabla_x)v(x) = f^\Omega(x), \quad x \in \Omega,$$

$$(18) \qquad \mathcal{B}^\Omega(x, \nabla_x)v(x) = g^\Omega(x), \quad x \in \partial\Omega\ .$$

As usual $C^{s,\alpha}(\Omega)$ denotes the Hölder space for $s \in \mathbb{N}_0 := \{0, 1, ...\}$, $\alpha \in (0,1)$, with the norm

$$\|z; C^{s,\alpha}(\Omega)\| = \sum_{k=0}^{s} \sup_\Omega |\nabla_x^k z(x)| + \sup_\Omega \sup_\Omega |x-y|^{-\alpha} |\nabla_x^s z(x) - \nabla_y^s z(y)|\ .$$

We associate with problem (17)–(18), for $l \in \mathbb{N} := \{1, 2, ...\}$, $\alpha \in (0,1)$, the linear and continuous mapping

$$(19) \qquad A^\Omega\ :\ C^{l+1,\alpha}(\Omega)^T \to C^{l-1,\alpha}(\Omega)^T \times \prod_{j=1}^{T} C^{l+1-r_j^\Omega,\alpha}(\partial\Omega) =: \mathcal{R}^{l,\alpha}C(\Omega, \partial\Omega)\ ,$$

where r_j^Ω stands for the order of the differential operators in the row $\mathcal{B}_j$, thus $r_j^\Omega = 1$ or $r_j^\Omega = 0$ depending on the definition (10)–(11).

Theorem 2.1 (cf. [45], [52]) *The ellipticity of the boundary value problem defined by operators* $\{\mathcal{L}, \mathcal{B}^\Omega\}$ *in* $\Omega \times \partial\Omega$ *follows from the polynomial property* (14). *The operator* (19) *is Fredholm of index zero, its kernel* $\ker A^\Omega$ *defined as the linear space of smooth solutions to the homogeneous problem* (17)–(18) *coincides with the linear space* $P(\Omega)$ *given by* (15). *There exists a solution in the space* $C^{l+1,\alpha}(\Omega)^T$ *to problem* (17)–(18) *with the right hand side* $\{f^\Omega, g^\Omega\} \in \mathcal{R}^{l,\alpha}C(\Omega, \partial\Omega)$, *if and only if the following compatibility condition is satisfied*

$$(20) \qquad (f^\Omega, p)_\Omega + (g^\Omega, \mathcal{T}^\Omega p)_{\partial\Omega} = 0 \quad \forall p \in P(\Omega)\ .$$

2.2 Weighted Hölder spaces and solvability of the second limit problem

The second limit problem is formally derived by replacing in (3)–(5) the *slow* variable x by the *fast* variable ξ from (1) and by setting $\varepsilon = 0$. By this inflation the set $\Omega(\varepsilon)$ becomes $\Omega_{\frac{1}{\varepsilon}} \setminus \overline{\omega}$, and the boundary of $\Omega_{\frac{1}{\varepsilon}}$ disappears at infinity with $\varepsilon \to 0^+$. The second limit problem takes the form

$$\mathcal{L}(\nabla_\xi)w(\xi) = f^\omega(\xi), \quad \xi \in G := \mathbb{R}^n \setminus \overline{\omega} , \tag{21}$$

$$\mathcal{B}^\omega(\xi, \nabla_\xi)w(\xi) = g^\omega(\xi), \quad \xi \in \partial G = \partial\omega . \tag{22}$$

In order to establish the solvability of (21)-(22) we introduce function spaces with weighted norms.

The general theory of elliptic problems in domains $G \subset \mathbb{R}^n$ with piecewise smooth boundaries ∂G (see, e.g., [43], [23]) is developed in the weighted spaces $V^s_\gamma(G)$ of Sobolev type as well as in the weighted spaces $\Lambda^{s,\alpha}_\gamma(G)$ of Hölder type.

These spaces are defined as the completion of the linear space $C_0^\infty(\overline{G})$ of arbitrarily smooth functions with compact supports in the following norms

$$\|z; V^s_\gamma(G)\| = \left(\sum_{k=0}^{s} \|\rho^{\gamma - s + k} \nabla^k_\xi z; L_2(G)\|^2 \right)^{\frac{1}{2}} , \tag{23}$$

$$\begin{aligned} \|z; \Lambda^{s,\alpha}_\beta(G)\| = &\sum_{k=0}^{s} \sup_{\xi \in G} \rho^{\beta - s - \alpha + k} |\nabla^k_\xi z(\xi)| \\ &+ \sup_{\xi \in G} \sup_{\{\eta \in G \,:\, 2|\xi - \eta| < |\xi|\}} \rho^\beta |\xi - \eta|^{-\alpha} |\nabla^s_\xi z(\xi) - \nabla^s_\eta z(\eta)| , \end{aligned} \tag{24}$$

respectively. Here, $\beta, \gamma \in \mathbb{R}$ are the weight indices, and $\rho = |\xi|$. Since $\mathcal{O} \in \omega$, the inequalities $0 < c_\omega \le \rho \le C_\omega$ hold on $\partial\omega$ and therefore, the space of traces on $\partial\omega$ of functions from the space $\Lambda^{s,\alpha}_\beta(G)$ coincides with the standard Hölder space $C^{s,\alpha}(\partial\omega)$.

The exponent λ of the function $\rho^\lambda = |\xi|^\lambda$ in (23) and (24) influences the behaviour of the function z for large $|\xi|$, at infinity.

The space $\Lambda^{s,\alpha}_\beta(\Omega)$ is obtained by the completion in the norm given below, of the space $C_0^\infty(\overline{\Omega} \setminus \mathcal{O})$ which contains smooth functions vanishing in the vicinity of the origin $\mathcal{O}$. The norm in $\Lambda^{s,\alpha}_\beta(\Omega)$ is defined by the same formula as (24) with G, ξ, ρ replaced by Ω, x, r, respectively, where $r = |x|$ is the distance to the origin $\mathcal{O}$,

$$\begin{aligned} \|w; \Lambda^{s,\alpha}_\beta(\Omega)\| = &\sum_{k=0}^{s} \sup_{x \in \Omega} r^{\beta - s - \alpha + k} |\nabla^k_x w(x)| + \\ &+ \sup_{x \in \Omega} \sup_{\{y \in \Omega \,:\, 2|x - y| < |x|\}} r^\beta |x - y|^{-\alpha} |\nabla^s_x w(x) - \nabla^s_y w(y)| . \end{aligned} \tag{25}$$

In the case of the space $\Lambda^{s,\alpha}_\beta(\Omega)$, the exponent λ of the weight function $r^\lambda = |x|^\lambda$ influences the behaviour of the function w for small r, i.e. in the vicinity of the origin.

We shall also use the Hölder space $\Lambda^{s,\alpha}_\beta(\Omega(\varepsilon))$ of functions defined in the domain $\Omega(\varepsilon) = \Omega \backslash \overline{\omega}_\varepsilon$. Since $C_\Omega > r > c_\omega \varepsilon > 0$ in $\Omega(\varepsilon)$, the norm $\|w; \Lambda^{s,\alpha}_\beta(\Omega(\varepsilon))\|$ is equivalent to the usual norm $\|w; C^{s,\alpha}(\Omega(\varepsilon))\|$. However, at least one of the equivalence constants $C_\pm$ in the inequalities,

$$C_-^{-1} \|z; C^{s,\alpha}(\Omega(\varepsilon))\| \le \|z; \Lambda^{s,\alpha}_\beta(\Omega(\varepsilon))\| \le C_+ \|z; C^{s,\alpha}(\Omega(\varepsilon))\| ,$$

necessarily depends on the small parameter ε and tends to infinity with $\varepsilon \to 0^+$.
The following theorem establishes the existence of a solution to the exterior problem (21), (22). By r_j^ω we denote the orders of the differential operators in the row $\mathcal{B}_j^o$.

Theorem 2.2 *Let $l \in \mathbb{N}, \alpha \in (0,1)$. The operator associated to problem* (21)–(22),

$$\mathcal{A}^\omega \ : \ \Lambda_\beta^{l+1,\alpha}(G)^T \to \Lambda_\beta^{l-1,\alpha}(G)^T \times \prod_{j=1}^{T} C^{l+1-r_j^\omega,\alpha}(\partial\omega) =: \mathcal{R}_\beta^{l,\alpha}\Lambda(G,\partial\omega) \ , \tag{26}$$

is an isomorphism if and only if

$$0 < \beta - l - 1 - \alpha < n - 2 \ . \tag{27}$$

Remark

Heuristic way of adjusting constraints (27) *can be described as follows : The space* $\Lambda_\beta^{l+1,\alpha}(G)^T$ *should contain fundamental solutions, with the decay* $O(r^{2-n})$ *at infinity, but the constant vectors should be excluded from the space. Both, the finiteness of the norm in the first case and unrestrictedness of supremum in the second case define exactly the lower and upper bounds in* (27)*. The same argument applies to constraints* (29) *for mapping* (28) *of problem* (21)-(22) *in the weighted Sobolev classes.*

On the other hand, the same constraints (27) *serve to assure the following feature of operator* (37) *of the first limit problem* (17)-(18) *in weighted Hölder classes : Fundamental solution belongs to, and the constant columns are excluded from the space* $\Lambda_\beta^{l+1,\alpha}(\Omega)^T$*. The change of inclusions to reverse when replacing the exterior domain G by the bounded domain Ω is justified by the fact that in such a case $r \to \infty$ becomes $r \to 0+$.*

Proof(of Theorem 2.2) In ([43]; §6.4), (see also [50], [52]) it was shown that

1° The operator associated to problem (21)–(22),

$$A^\omega \ : \ V_\gamma^{s+1}(G)^T \to V_\gamma^{s-1}(G)^T \times \prod_{j=1}^{T} H^{s-r_j^\omega+\frac{1}{2}}(\partial\omega) =: \mathcal{R}_\gamma^s V(G,\partial\omega) \ , \tag{28}$$

is an isomorphism if and only if

$$-\frac{n}{2} + 1 < \gamma - s < \frac{n}{2} - 1 \ , \tag{29}$$

where $H^{s-r_j^\omega+\frac{1}{2}}(\partial\omega)$ is a Sobolev–Slobodetskii space.

To check up the declared properties of operator (26) we will use the relations between operators of boundary value problems in the scales of the weighted Sobolev spaces and the weighted Hölder spaces obtained in [31] (see also [43] §3.6, §4.1, [23]).

To this end we also need the following facts which were established in [31] and which were presented in [43] as well.

2° (Theorems 3.6.11, 4.1.8 in [43]). The linear mapping $\mathcal{A}^\omega$ defined by (26) is Fredholm if and only if the linear mapping A^ω defined by (28) with the weight index

$$\gamma = \beta - l - \alpha + s - \frac{n}{2} \tag{30}$$

is Fredholm.

3° (Theorems 1.1.7, 3.6.11, 4.1.8 in [43]). If the operator $\mathcal{A}^\omega$ in (26) is Fredholm, then the following inequality holds

$$\|w;\Lambda_\beta^{l+1,\alpha}(\Omega)\| \le c(\|\{\mathcal{L}w,\mathcal{B}^\omega w\};\mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega,\partial\Omega)\| + \|w;L_2(\mathbb{B}_R\setminus\omega)\|) \ , \tag{31}$$

for any function $w\in\Lambda_\beta^{l+1,\alpha}(\Omega)^T$, where $\mathbb{B}_R=\{\xi \ : \ |\xi|<R\}$ and $R>1$, i.e. $\omega\subset\mathbb{B}_R$.

It is easy to see that the transformation $\beta\mapsto\gamma$ defined by (30) maps the interval (27) onto the interval (29). Therefore, in view of **2°** and **1°**, the operator (26) is Fredholm if condition (27) is satisfied.

Let us denote by $u\in\Lambda_\beta^{l+1,\alpha}(G)$ the solution to problem (21)–(22) with the right hand side $\{f,g\}\in\mathcal{R}_\beta^{l,\alpha}\Lambda(G,\partial\omega)$. Then $u\in V_{\gamma-\mu}^{s+1}(G)^T$ for any $s\le l$ and $\mu>0$, with γ defined by (30). Besides, there holds the estimate

$$\begin{aligned} \|w;V_{\gamma-\mu}^{s+1}(G)\|^2 &= \sum_{k=0}^{s+1}\int_G \rho^{2(\gamma-\mu-s-1+k)}|\nabla_\xi^k w(\xi)|^2 d\xi \\ \le \sum_{k=0}^{s+1}&\left[\sup_G\left\{\rho^{\beta-l-1-\alpha+k}|\nabla_\xi^k w(\xi)|\right\}\right]^2\int_G\rho^{2(\gamma-\mu-s+\alpha-\beta)}d\xi \\ &\le C\|w;\Lambda_\beta^{l+1,\alpha}(G)\|^2 \ . \end{aligned}\tag{32}$$

The last integral over G converges since $\gamma-\mu-s+\alpha-\beta=-\frac{n}{2}-\mu$ and $\mu>0$.

We select μ small enough, so that $\gamma-\mu$ satisfies (29), thus we can write the upper bound given by **1°** as follows,

$$\|u;V_{\gamma-\mu}^{s+1}(G)\|\le c\|\{f,g\};\mathcal{R}_{\gamma-\mu}^s V(G,\partial\omega)\|\le C\|\{f,g\};\mathcal{R}_\beta^{l,\alpha}\Lambda(G,\partial\omega)\| \ . \tag{33}$$

The second inequality in (33) is obtained by the same argument as in (32).
The first inequality in (33) implies the uniqueness of the solution, i.e. the equality $\dim\ker\mathcal{A}^\omega=0$ since the operator A^ω in (28) is an isomorphism.

For indices $\varsigma>l+\alpha+\frac{n}{2}$ and $\varkappa=\beta-l-\alpha+\varsigma-\frac{n}{2}$,

$$V_\varkappa^{\varsigma+1}(G)\subset\Lambda_\beta^{l+1,\alpha}(G) \ ,$$

we refer the reader to [32], ([43], §6.6) for the simplest proof of the above embedding; see also [6]. Let

$$\{f,g\}\in C_0^\infty(\overline{G})^T\times C^\infty(\partial\omega)^T\subset\mathcal{R}_\varkappa^\varsigma V(G,\partial\omega) \ . \tag{34}$$

Since $\varsigma,\varkappa$ are related by (29), in view of (27), **1°** assures the existence of a solution w to (21)–(22) such that

$$w\in V_\varkappa^{\varsigma+1}(G)^T\subset\Lambda_\beta^{l+1,\alpha}(G)^T \ . \tag{35}$$

On the other hand, there exists a solution $w'\in V_{\gamma-\mu}^{s+1}(G)^T$ to (21)–(22), since in view of (34) the right hand side of (21)–(22) is in $\mathcal{R}_\sigma^s V(G,\partial\omega)$ for any s,σ. By the embeddings $V_\varkappa^{\varsigma+1}(G)\subset V_{\varkappa-\varsigma+s}^{s+1}(G)=V_\gamma^{s+1}(G)\subset V_{\gamma-\mu}^{s+1}(G)$, the uniqueness of the solution established in **1°** implies that $w=w'$ with the estimate (33) for w. Furthermore, **3°** applied to the solution of (35) gives the relation

$$\begin{aligned}\|w;\Lambda_\beta^{l+1,\alpha}(G)\| &\le c(\|\{f,g\};\mathcal{R}_\beta^l\Lambda(G,\partial\omega)\|+\|w;L_2(\mathbb{B}_R\setminus\omega)\|)\\ &\le c(1+Cc_R)\|\{f,g\};\mathcal{R}_\beta^l\Lambda(G,\partial\omega)\|,\end{aligned}\tag{36}$$

where the latter inequality results from (33) combined with the inequality

$$\|w; L_2(\mathbb{B}_R \setminus \omega)\| \leq c_R \|w; V^{s+1}_{\gamma-\mu}(G)\| \ .$$

Theorem 2.2 follows in the standard way by completing the space $C_0^\infty(\overline{G})^T \times C^\infty(\partial\omega)^T$ in the norm $\|\cdot; \mathcal{R}^l_\beta \Lambda(G, \partial\omega)\|$, which means that solutions which satisfy (36) do exist for all right hand sides $\{f, g\} \in \mathcal{R}^l_\beta \Lambda(G, \partial\omega)$. □

2.3 Solvability of the first limit problem in weighted Hölder spaces

The following result which is useful for further applications is established for the operator of the boundary value problem (17)–(18),

$$(37) \qquad \mathcal{A}^\Omega \ : \ \Lambda^{l+1,\alpha}_\beta(\Omega)^T \to \Lambda^{l-1,\alpha}_\beta(\Omega)^T \times \prod_{j=1}^{T} C^{l+1-r^\Omega_j,\alpha}(\partial\Omega) =: \mathcal{R}^{l,\alpha}_\beta \Lambda(\Omega, \partial\Omega) \ .$$

The proof of this result is similar to that of Theorem 2.2 and, thus is omitted here. The properties of the operator $\mathcal{A}^\Omega$ in weighted Sobolev spaces required for the proof are given e.g., in ([43], § 6.4), [52].

Theorem 2.3 *If condition* (27) *is satisfied, then the operator* (37) *is Fredholm of index zero,* $\ker\mathcal{A}^\Omega = P(\Omega)$ *and* $\mathrm{coker}\mathcal{A}^\Omega = \{(p|_\Omega \, , \mathcal{T}^\Omega p|_{\partial\Omega}) \ : \ p \in P(\Omega)\}$. *In other words, the only solutions to the homogeneous problem* (17)–(18) *in the space* $\Lambda^{l+1,\alpha}_\beta(\Omega)^T$ *are polynomials from* $P(\Omega)$. *The compability conditions for problem* (17)–(18) *with right hand side* $\{f^\Omega, g^\Omega\} \in \mathcal{R}^{l,\alpha}_\beta \Lambda(\Omega, \partial\Omega)$ *are given by* (20).

2.4 Solvability of perturbed problems and uniform estimates of solutions

Problem (3)–(5) is elliptic by the polynomial property (14). In spite of the presence of the small parameter ε, the following general result [45], [52] applies to (3)–(5).

Proposition 2.1 *The operator associated with boundary value problem* (3)–(5),

$$(38) \qquad C^{l+1,\alpha}(\Omega(\varepsilon))^T \to C^{l-1,\alpha}(\Omega(\varepsilon))^T \times \prod_{j=1}^{T} C^{l+1-r^\Omega_j,\alpha}(\partial\Omega) \times \prod_{j=1}^{T} C^{l+1-r^\omega_j,\alpha}(\partial\omega_\varepsilon)$$

is Fredholm with index zero, its kernel is given by the intersection $P(\Omega) \cap P(\omega)$. *The compability conditions are necessary and sufficient for the existence of a solution to problem* (3)–(5),

$$(39) \qquad (f, p)_{\Omega(\varepsilon)} + (g, \mathcal{T}^\Omega p)_{\partial\Omega} + (g^\omega, \mathcal{T}^\omega p)_{\partial\omega_\varepsilon} = 0 \quad \forall p \in P(\Omega) \cap P(\omega) \ .$$

We recall that the estimates in standard Hölder norms established in Proposition 2.1 for the solution $u(\varepsilon, \cdot)$ to (3)–(5) cannot be asymptotically sharp since the standard Hölder norms are not adapted to reflect the dependence of the solution $u(\varepsilon, \cdot)$ with respect to the small geometrical parameter. Furthermore, it will become clear after the proof of Theorem 2.4 is completed, that even in the case of the trivial kernel, $P(\Omega) = 0$, which implies the uniqueness of the solution to the limit problem (17)–(18), the norm of the inverse of the mapping (38) is unbounded, i.e.

it tends to infinity, as $\varepsilon \to 0^+$. Therefore, instead of the mapping (38), the following operator associated with problem (46)–(48) is considered in Theorem 2.4,

$$\mathcal{A}^\epsilon \ : \ \Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))^T \to \mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon) =: \tag{40}$$
$$\Lambda_\beta^{l-1,\alpha}(\Omega(\varepsilon))^T \times \prod_{j=1}^{T} \Lambda_\beta^{l+1-r_j^\Omega,\alpha}(\partial\Omega) \times \prod_{j=1}^{T} \Lambda_\beta^{l+1-r_j^\omega,\alpha}(\partial\omega_\varepsilon) \ .$$

The norm of the operator (40) is bounded, uniformly with respect to $\varepsilon \in (0,1]$. Furthermore, under the additional assumption

$$P(\Omega) \subset P(\omega) \ , \tag{41}$$

the operator $\mathcal{A}^\epsilon$ restricted to the complement of its kernel in the space $\Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))^T$ is invertible, with the norm of inverse uniformly bounded with respect to $\varepsilon \in (0,1]$, provided that l, α and β fulfill relation (27).

Let us point out the following facts which are used in the analysis of problem (3)–(5):

1° The norm $\|\cdot; \Lambda_\beta^{l+1-r_j^\omega,\alpha}(\partial\Omega)\|$ in the space $C^{l+1-r_j^\omega,\alpha}(\partial\Omega)$ is equivalent, with respect to ε, to the standard Hölder norm without weights. Moreover, for any function g, the norm $\|g; \Lambda_\beta^{l+1-r_j^\omega,\alpha}(\partial\omega_\varepsilon)\|$ is *uniformly* equivalent to the following norm, dependent on the small parameter ε,

$$\sum_{k=0}^{l+1-r_j^\omega} \varepsilon^{\beta-l-1+r_j^\omega+k-\alpha} \sup_{x\in\partial\omega_\varepsilon} |\nabla_s^k g(x)| \tag{42}$$
$$+\varepsilon^\beta \sup_{x,y\in\partial\omega_\varepsilon} |x-y|^{-\alpha} |\nabla_s^{l+1-r_j^\omega} g(x) - \nabla_s^{l+1-r_j^\omega} g(y)| \ .$$

In the sequel we shall use for function spaces the notation introduced in (40).

2° If assumption (41) is not satisfied, the norm of the inverse operator $(\mathcal{A}^\epsilon)^{-1}$ grows with $\varepsilon \to 0^+$. This fact can be deduced from the general results of [35] (see also chapter 4 in books [37], [38]). Avoiding to discuss this situation in general, we mention here that in [7], for the Kirchoff plate problem, a finite dimensional but unbounded (with $\varepsilon \to 0+$) component of the inverse operator $(\mathcal{A}^\epsilon)^{-1}$ is extracted, such that the remainder becomes uniformly bounded with respect to $\varepsilon \in (0,1]$.

In order to simplify the derivation of uniform estimates, boundary value problem (3)–(5) is reformulated in such a way that the resulting problem enjoys a unique solution. From Proposition 2.1 it follows that the dimensions of *kernel* and *co-kernel* of operator (38) coincide. Therefore, the new formulation of the problem can be obtained by direct annulation of the kernel and co-kernel of the linear mapping. To this end, a basis $\{p^1, ..., p^\ell\}$ in the linear space $P(\Omega)$ of the dimension $\ell :=\dim P(\Omega)$ is introduced, and the functions $\{h^1, ..., h^\ell\}$ bi-orthogonal for the basis are selected,

$$(p^k, h^j)_\Omega = \delta_{j,k} \ , \quad j,k = 1, ..., \ell \ . \tag{43}$$

Observing that, by Theorems 2.1, 2.3 and Proposition 2.1 (see (41)), the orthogonality conditions

$$(v, h^j)_\Omega = 0 \ , \quad j = 1, ..., \ell; \tag{44}$$
$$(u, h^j)_{\Omega(\varepsilon)} = 0 \ , \quad j = 1, ..., \ell, \tag{45}$$

provide the uniqueness of solutions to the problems (17)–(18) and (3)–(5), respectively. We replace problem (3)–(5) by the problem

$$\mathcal{L}(\nabla_x)u(\varepsilon,x)+\sum_{j=1}^{\ell}h^j(x)(u,h^j)_{\Omega(\varepsilon)}=f(\varepsilon,x),\quad x\in\Omega(\varepsilon)\ , \tag{46}$$

$$\mathcal{B}^{\Omega}(x,\nabla_x)u(\varepsilon,x)=g(\varepsilon,x),\quad x\in\partial\Omega\ , \tag{47}$$

$$\mathcal{B}^{\omega}(\varepsilon^{-1}x,\nabla_x)u(\varepsilon,x)=g^{\omega}(\varepsilon,x),\quad x\in\partial\omega_\varepsilon\ . \tag{48}$$

The above problem is not any more purely partial differential equations, however the additional terms are compact. Therefore, the associated linear operator considered as mapping (38) remains Fredholm.

We point out that due to the uniqueness of a solution to problem (46)-(48) we can forget about the small correction $\widehat{g}$ in (12). All along the asymptotic analysis performed in the sequel we suppose that $g(\varepsilon,x)=g^{\Omega}(x)$. We return to the correction $\widehat{g}$ only when dealing with the formulation of final result for problem (3)-(5).

Lemma 2.1 **1**) *Operator* (38) *associated with the boundary value problem* (46)–(48) *is an isomorphism.*

2) *If the orthogonality condition* (39) *is fulfilled, then any solution to* (46)–(48) *coincides with the solution to* (3)–(5) *subject to* (45).

Proof

1) Assuming that conditions (43) as well as (45) and (39) are satisfied, we introduce the orthogonal decomposition, with respect to the scalar product in $L_2(\Omega(\varepsilon))$,

$$C^{l+1,\alpha}(\Omega(\varepsilon))^T=D_\perp\oplus\mathfrak{L}(h^1,...,h^\ell), \tag{49}$$

$$\mathcal{R}^{l+1,\alpha}C(\Omega(\varepsilon),\partial\Omega,\partial\omega_\varepsilon)=R_\perp\oplus\{\mathfrak{L}(h^1,...,h^\ell),0,0\}, \tag{50}$$

where $\mathfrak{L}(h^1,...,h^\ell)$ denotes the linear hull of the set $h^1,...,h^\ell$. From Proposition 2.1 it follows that the restriction $D_\perp\mapsto R_\perp$ of the operator associated to problem (3)–(5) is an isomorphism. The operator associated to problem (46)–(48) is an extension of such a restriction, and in addition its finite dimensional component

$$\mathfrak{L}(h^1,...,h^\ell)\ni u\mapsto\left\{\sum_{j=1}^{\ell}h^j(u,h^j)_{\Omega(\varepsilon)},0,0\right\}\in\{\mathfrak{L}(h^1,...,h^\ell),0,0\}$$

turns out to be the natural identification of the last subspaces in (50). This completes the proof of **1**).

2) We write the Green's formula (7) for the solution u to problem (46)–(48) and for a polynomial p^k, which leads to the equality

$$\left(f-\sum_{j=1}^{\ell}h^j(u,h^j)_{\Omega(\varepsilon)},p^k\right)_{\Omega(\varepsilon)}+(g^{\Omega},\mathcal{T}^{\Omega}p^k)_{\partial\Omega}+(g^{\omega},\mathcal{T}^{\omega}p^k)_{\partial\omega_\varepsilon}=0\ .$$

Using bi-orthogonality (43) and condition (39) the above equality becomes $(u,h^k)_{\Omega(\varepsilon)}=0$, which completes the proof of **2**). □

Theorem 2.4 *Let the indices of regularity l, α and the weight index β satisfy the inequalities (29). Then the norm of the operator $(\mathcal{A}_h^\varepsilon)^{-1}$, the inverse of the mapping associated to problem (46)–(48),*

$$(51) \qquad \mathcal{A}_h^\varepsilon \ : \ \Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))^T \to \mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon),$$

is bounded by a constant independent of $\varepsilon \in (0,1]$.

Proof
We construct an almost–inverse R_h^ε of the operator $\mathcal{A}_h^\varepsilon$,

$$(52) \qquad R_h^\varepsilon \ : \ \mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon) \to \Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))^T$$

i.e. such an operator that for some $\delta > 0$ the following inequalities hold true,

$$(53) \qquad \|R_h^\varepsilon; \mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon) \to \Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))^T\| \le C \ ,$$

$$(54) \qquad \|\mathcal{A}_h^\varepsilon R_h^\varepsilon - \mathbb{I} \ ; \mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon) \hookleftarrow \| \le C_\delta \varepsilon^\delta$$

with the constants C, C_δ, independent of $\varepsilon \in (0, \varepsilon_0]$. Here $\mathbb{I}$ denotes the identity operator. In this way we show that the operator $\mathcal{A}_h^\varepsilon R_h^\varepsilon$ is invertible and we have $(\mathcal{A}_h^\varepsilon)^{-1} = R_h^\varepsilon \, (\mathcal{A}_h^\varepsilon R_h^\varepsilon)^{-1}$. The norm of the operator $(\mathcal{A}_h^\varepsilon)^{-1}$ is bounded, in view of (53)–(54), by the product of the norms of operators R_h^ε and $(\mathcal{A}_h^\varepsilon R_h^\varepsilon)^{-1}$.

Using the continuity of the norm of the operator $(\mathcal{A}_h^\varepsilon)^{-1}$ with respect to the parameter $\varepsilon > 0$ the result can be extended for $\varepsilon \in [\varepsilon_0, 1]$.

Let δ_0 be sufficiently small, such that inequality (27) holds for the index β replaced with $\beta \pm 2\delta_0$, and for l and α remaining unchanged. For given $\{f, g, g^\omega\} \in \mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon)$ we introduce the elements,

$$(55) \qquad \begin{aligned} f^1(\varepsilon, x) = (1 - \chi_{\frac{1}{2}}(\varepsilon, x))f(\varepsilon, x) \ , \quad g^1(\varepsilon, x) = g(\varepsilon, x); \\ f^2(\varepsilon, \xi) = \varepsilon^2 \chi_{\frac{1}{2}}(\varepsilon, \varepsilon\xi) f(\varepsilon, x) \ , \\ g^2 = (g_1^2, ..., g_T^2) \ , \quad g_j^2(\varepsilon, \xi) = \varepsilon^{r_j^\omega} g_j^\omega(\varepsilon, \xi), \end{aligned}$$

where, with a cutoff function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$, we put

$$(56) \qquad \chi_\alpha(\varepsilon, \xi) = \chi(\varepsilon^{-\alpha} x) \ , \ \alpha \in [0, 1] \ .$$

Clearly

$$(57) \qquad f(\varepsilon, x) = f^1(\varepsilon, x) + \varepsilon^{-2} f^2(\varepsilon, \varepsilon^{-1} x) \ .$$

The inclusions $f^1 \in \Lambda_{\beta-2\delta}^{l-1,\alpha}(\Omega)$, $g_j^1 \in \Lambda_{\beta-2\delta}^{l+1-r_j^\Omega,\alpha}(\partial\Omega)$, $f^2 \in \Lambda_{\beta+2\delta}^{l-1,\alpha}(G)$, $g_j^2 \in \Lambda_{\beta+2\delta}^{l+1-r_j^\omega,\alpha}(\partial\omega)$ hold true with any $\delta \in [0, \delta_0]$ since $f^1 = 0$ in a neighbourhood of the origin $x = \mathcal{O}$ and $f^2 = 0$ outside a ball.
We have for f^1,

$$(58) \qquad \begin{aligned} \|f^1; \Lambda_{\beta-2\delta}^{l-1,\alpha}(\Omega)\| = \|(1 - \chi_{\frac{1}{2}})f; \Lambda_{\beta-2\delta}^{l-1,\alpha}(\Omega \setminus \mathbb{B}_{\sqrt{\varepsilon}})\| \le \\ \le \varepsilon^{-\delta}\|(1 - \chi_{\frac{1}{2}})f; \Lambda_\beta^{l-1,\alpha}(\Omega(\varepsilon))\| \le C_\chi \varepsilon^{-\delta}\|f; \Lambda_\beta^{l-1,\alpha}(\Omega(\varepsilon))\| \ , \end{aligned}$$

and for g_j^1,

(59) $$\|g_j^1;\Lambda_{\beta-2\delta}^{l+1-r_j^\Omega,\alpha}(\partial\Omega)\| \le c\|g_j;\Lambda_\beta^{l+1-r_j^\Omega,\alpha}(\partial\Omega)\| \ .$$

Finally for f^2,

(60) $$\begin{aligned}\|f^2;\Lambda_{\beta+2\delta}^{l-1,\alpha}(G)\| &= \varepsilon^2\|\chi_{\frac12}(\varepsilon,\varepsilon\cdot)f(\varepsilon,\varepsilon\cdot);\Lambda_{\beta+2\delta}^{l-1,\alpha}(\mathbb{B}_{\frac{2}{\sqrt{\varepsilon}}}\setminus\omega)\| \le \\ &\le (4\varepsilon)^{-\delta}\varepsilon^2\|\chi_{\frac12}(\varepsilon,\varepsilon\cdot)f(\varepsilon,\varepsilon\cdot);\Lambda_{\beta}^{l-1,\alpha}(\mathbb{B}_{\frac{2}{\sqrt{\varepsilon}}}\setminus\omega)\| = \\ &= (4\varepsilon)^{-\delta}\varepsilon^2\varepsilon^{-(\beta-l+1-\alpha)}\|\chi_{\frac12}f;\Lambda_{\beta}^{l-1,\alpha}(\mathbb{B}_{\frac{2}{\sqrt{\varepsilon}}}\setminus\omega)\| \le \\ & c\varepsilon^{-\delta}\varepsilon^{-(\beta-l+1-\alpha)}\|f;\Lambda_{\beta}^{l-1,\alpha}(\Omega(\varepsilon))\| \ ,\end{aligned}$$

and for g_j^2,

(61) $$\begin{aligned}&\|g_j^2;\Lambda_{\beta+2\delta}^{l+1-r_j^\omega,\alpha}(\partial\omega)\| \le c\|g_j^2;\Lambda_{\beta}^{l+1-r_j^\omega,\alpha}(\partial\omega)\| = \\ &= c\varepsilon^{r_j^\omega}\|g_j^\omega(\varepsilon\cdot);\Lambda_{\beta}^{l+1-r_j^\omega,\alpha}(\partial\omega)\| = c\varepsilon^{-(\beta-l-1-\alpha)}\|g_j^\omega;\Lambda_{\beta}^{l+1-r_j^\omega,\alpha}(\partial\omega_\varepsilon)\| \ .\end{aligned}$$

The following properties are used, besides the definitions of weighted spaces, in order to establish (58)-(61),

1° $r=|x|\ge\varepsilon^{\frac12}$ on the support of the cutoff function $1-\chi_{\frac12}(\varepsilon,x)$;

2° $\rho=|\xi|\le 2\varepsilon^{\frac12}$ on the support of the cutoff function $\chi_{\frac12}(\varepsilon,\varepsilon\xi)$;

3° the change of variables $\xi\mapsto x=\varepsilon\xi$ transforms norms (23)–(24) into the expression $\varepsilon^{-(\beta-s-\alpha)}\|x\mapsto z(\varepsilon^{-1}x);\Lambda_\beta^{s,\alpha}(G_\varepsilon)\|$, where $G_\varepsilon=\mathbb{R}^n\setminus\omega_\varepsilon$.

Remark *The properties* **1**° *and* **2**° *allow for changing the weight indices of function spaces, and to include the appropriate multipliers in the form of the powers of* ε. *Due to property* **3**° *the additional multiplier* $\varepsilon^{-(\beta-l+1-\alpha)}$ *appears in* (60)-(61).

We return to the proof of Theorem 2.4. Let w be the solution of the second limit problem (21)–(22) with the right hand side $\{f^2,g^2\}$, such a solution exists by Theorem 2.3. Since the support of f^2 is compact and $\Lambda_\beta^{l+1,\alpha}(G)^T\subset\Lambda_{\beta+2\delta}^{l+1,\alpha}(G)^T$, the unique solution w belongs to the spaces $\Lambda_{\beta+2\delta}^{l+1,\alpha}(G)^T$ for all $\delta\in[0,\delta_0]$. An application of (60)–(61) leads to the estimate:

(62) $$\|w;\Lambda_{\beta+2\delta}^{l+1,\alpha}(G)^T\| \le c\|\{f^2,g^2\};\mathcal{R}_{\beta+2\delta}^{l,\alpha}\Lambda(G,\partial\omega)\| \le c\varepsilon^{-\delta}\varepsilon^{-(\beta-l-1-\alpha)}\mathbf{N}\,,$$

where

$$\mathbf{N}:=\|\{f,g,g^\omega\};\mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon),\partial\Omega,\partial\omega_\varepsilon)\| \ .$$

We denote by v a solution to the boundary value problem

(63) $$\mathcal{L}v+\sum_{j=1}^{\ell}h^j(v,h^j)_\Omega=f^1 \quad \text{in } \Omega \ ,$$

(64) $$\mathcal{B}^\Omega v=g^1 \quad \text{on } \partial\Omega \ ,$$

which is obtained by the perturbation of the first limit problem using the same finite dimensional operator as in (46). Repeating the arguments from the proof of Lemma 2.1(1) we can extend Theorem 2.3 to problem (63)–(64). In this way, the existence in the space $\Lambda_{\beta-2\delta}^{l+1,\alpha}(\Omega)^T$ of the unique solution v to (63)–(64) is established for any $\delta \in [0, \delta_0]$. In particular, definition (55) implies that the support of f^1 is disjoint from the origin $\mathcal{O}$. Using (58)–(59), the estimate for the solution to (63)–(64) is derived

$$\|v; \Lambda^{l+1,\alpha}_{\beta-2\delta}(\Omega)\| \leq c\|\{f^1, g^1\}; \mathcal{R}^{l,\alpha}_{\beta-2\delta}\Lambda(\Omega, \partial\Omega)\| \leq c\varepsilon^{-\delta}\mathbf{N} \ . \tag{65}$$

Following [35] (see also [37]) we construct an approximation to the solution to (3)–(5),

$$U = R^\varepsilon_h\{f, g, g^\omega\} \ , \tag{66}$$

by *gluing* the solutions v and w of the first and the second limit problems :

$$U(\varepsilon, x) = (1 - \chi_1(\varepsilon, x))v(\varepsilon, x) + \chi_0(\varepsilon, x)w(\varepsilon, \varepsilon^{-1}x) \tag{67}$$

Remark *The supports of cutoff functions overlap each other:*

$$\begin{gathered} 1 - \chi_1(\varepsilon, x) = 1 \quad \text{for } |x| \geq 2\varepsilon \ , \quad \text{but } 1 - \chi_1(\varepsilon, x) = 0 \quad \text{on } \partial\omega_\varepsilon \ , \\ \chi_0(\varepsilon, x) = 1 \quad \text{for } |x| \leq 1 \quad \text{or } |\xi| \leq \varepsilon^{-1} \ , \quad \text{but } \chi_0(\varepsilon, x) = 0 \quad \text{on } \partial\Omega \ . \end{gathered} \tag{68}$$

Using formulae (62) and (65) with $\delta = 0$ leads to

$$\begin{aligned} \|U; \Lambda^{l+1,\alpha}_\beta(\Omega(\varepsilon))\| &\leq c(\|(1-\chi_1)v; \Lambda^{l+1,\alpha}_\beta(\Omega)\| + \|\chi_0(\varepsilon,\cdot)w(\varepsilon,\cdot); \Lambda^{l+1,\alpha}_\beta(G_\varepsilon)\| \\ &\leq c(\|v; \Lambda^{l+1,\alpha}_\beta(\Omega)\| + \varepsilon^{+(\beta-l-1-\alpha)}\|w; \Lambda^{l+1,\alpha}_\beta(G)\|) \leq \\ &\leq c(1 + \varepsilon^{+(\beta-l-1-\alpha)}\varepsilon^{-(\beta-l-1-\alpha)})\mathbf{N} = 2c\mathbf{N} \ . \end{aligned}$$

Therefore, the linear operator R^ε_h defined by (66), satisfies (52) and (53). Now, we estimate the norm of the operator $\mathcal{A}^\varepsilon_h R^\varepsilon_h - \mathbb{I}$, or equivalently, the norm of the difference $\mathcal{A}^\varepsilon_h U - \{f, g, g^\omega\}$ in the space $\mathcal{R}^{l,\alpha}_\beta\Lambda(\Omega, \partial\Omega)$, i.e., the discrepancy of the approximate solution U relative to problem (53)–(54). In view of (68), (64) and (55), boundary condition (47) is fulfilled. It can be also verified that boundary condition (48) is satisfied. We return to equation (46), which is rewritten as follows,

$$\begin{aligned} \mathcal{L}U + \sum_{j=1}^{\ell} h^j(U, h^j)_{\Omega(\varepsilon)} - f &= \\ = \mathcal{L}((1-\chi_1)v + \chi_0 w) + \sum_{j=1}^{\ell} h^j((1-\chi_1)v + \chi_0 w, h^j)_{\Omega(\varepsilon)} - f &= \\ = \underbrace{(1-\chi_1)\mathcal{L}v + \chi_0\mathcal{L}w - f + \sum_{j=1}^{\ell} h^j(v, h^j)_\Omega}_{\mathbf{I}=0} - \underbrace{[\mathcal{L}, \chi_1]v + [\mathcal{L}, \chi_0]w}_{\mathbf{III}} & \\ + \underbrace{\sum_{j=1}^{\ell} h^j\left\{((1-\chi_1)v + \chi_0 w, h^j)_{\Omega(\varepsilon)} - (v, h^j)_\Omega\right\}}_{\mathbf{II}} , & \end{aligned} \tag{69}$$

here we denote by $[A, B] = AB - BA$ the commutator of operators A and B. Some comments on the underbraced terms are listed below.

- **(I)** By (55), (57), (6) and (63), taking into account the identities for cutoff functions, $(1-\chi_1)(1-\chi_{\frac{1}{2}}) = (1-\chi_{\frac{1}{2}})$ and $\chi_1\chi_{\frac{1}{2}} = \chi_{\frac{1}{2}}$, we obtain

$$(1-\chi_1)\mathcal{L}v + \sum_{j=1}^{\ell} h^j(v,h^j)_\Omega + \chi_0\mathcal{L}w = f^1 + \varepsilon^{-2}f^2 = f\ .$$

  Thus the term **(I)** vanishes.

- **(II)** Since $\omega_\varepsilon \subset \mathbb{B}_1$ and $h^j = 0$ in the ball $\mathbb{B}_1$, we have

$$(U,h^j)_{\Omega(\varepsilon)} = (v,h^j)_\Omega + (\chi_0 w, h^j)_{\Omega\setminus\mathbb{B}_1}\ .$$

  The last term is bounded as follows

$$|(\chi_0 w, h^j)_{\Omega\setminus\mathbb{B}_1}| \le c\sup_G\{\rho^{\beta-l-1-\alpha+2\delta}|w(\xi)|\}\int_1^{\operatorname{diam}\Omega}\left(\frac{\varepsilon}{r}\right)^{\beta-l-1-\alpha+2\delta} r^n dr \le$$
$$\le c\varepsilon^{-\delta}\mathbf{N}\,\underbrace{\varepsilon^{\beta-1-l-\alpha}}_{\le 1}\varepsilon^{2\delta} \le c\varepsilon^{\delta}\mathbf{N}\ ,$$

  where the exponent $\beta-1-l-\alpha$ is positive by (27). In this way, an upper bound for the norm $\Lambda^{l-1,\alpha}_\beta(\Omega(\varepsilon))$ of the underbraced term **(II)** is determined.

- **(III)** Let us consider the commutators $[\mathcal{L},\chi_1]$ and $[\mathcal{L},\chi_0]$. The supports of coefficients of the differential operator $[\mathcal{L},\chi_1]$ are included into the annulus $\{x\ :\ \varepsilon \le r = |x| \le 2\}$, therefore by (65), (58),

$$\|[\mathcal{L},\chi_1]v;\Lambda^{l-1,\alpha}_\beta(\Omega(\varepsilon))\| \le c\|v;\Lambda^{l-1,\alpha}_\beta(\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon)\| \le$$
$$\le c\varepsilon^{2\delta}\|v;\Lambda^{l-1,\alpha}_{\beta-2\delta}(\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon)\| \le c\varepsilon^{2\delta}\|v;\Lambda^{l-1,\alpha}_{\beta-2\delta}(\Omega)\| \le c\varepsilon^{\delta}\mathbf{N}\ .$$

  In a similar way, the supports of coefficients of the differential operator $[\mathcal{L},\chi_0]$ are included in the annulus $\{x\ :\ 1<r<2\}$, or equivalently $\{\xi\ :\ \varepsilon^{-1}<\rho<2\varepsilon^{-1}\}$, therefore by (62), (60),

$$\|[\mathcal{L},\chi_0]w;\Lambda^{l-1,\alpha}_\beta(\Omega(\varepsilon))\| = \|[\mathcal{L}(\nabla_x),\chi_0(\varepsilon,\cdot)]w(\varepsilon,\varepsilon^{-1}\cdot);\Lambda^{l-1,\alpha}_\beta(\mathbb{B}_2\setminus\mathbb{B}_1)\| =$$
$$= \varepsilon^{-2}\varepsilon^{\beta-l-1-\alpha}\|[\mathcal{L}(\nabla_\xi),\chi_0(\varepsilon,\varepsilon\cdot)]w(\varepsilon,\cdot);\Lambda^{l-1,\alpha}_\beta(\mathbb{B}_{\frac{2}{\varepsilon}}\setminus\mathbb{B}_{\frac{1}{\varepsilon}})\| \le$$
$$\le c\varepsilon^{\beta-l-1-\alpha}\|w;\Lambda^{l+1,\alpha}_\beta(\mathbb{B}_{\frac{2}{\varepsilon}}\setminus\mathbb{B}_{\frac{1}{\varepsilon}})\| \le c\varepsilon^{\beta-l-1-\alpha}\varepsilon^{2\delta}\|w;\Lambda^{l+1,\alpha}_{\beta-2\delta}(\mathbb{B}_{\frac{2}{\varepsilon}}\setminus\mathbb{B}_{\frac{1}{\varepsilon}})\| \le$$
$$\le c\varepsilon^{\beta-l-1-\alpha}\varepsilon^{2\delta}\varepsilon^{-\delta}\varepsilon^{-(\beta-l-1-\alpha)}\mathbf{N} \le c\varepsilon^{\delta}\mathbf{N}\ .$$

Finally, we combine the above estimate to derive

$$\|\mathcal{L}U + \sum_{j=1}^{\ell} h^j(U,h^j)_{\Omega(\varepsilon)} - f;\Lambda^{l-1,\alpha}_\beta(\Omega(\varepsilon))\| \le$$
$$\le c\varepsilon^{\delta}\|\{f,g,g^\omega\};\mathcal{R}^{l,\alpha}_\beta\Lambda(\Omega(\varepsilon),\partial\Omega,\partial\omega_\varepsilon)\|\ .$$

In order to complete the proof, for any $\delta \in (0,\delta_0]$, we select sufficiently small $\varepsilon \le \varepsilon(\delta_0)$ such that (54) follows. Therefore, the operator R^ε_h is the almost–inverse of the operator $\mathcal{A}^\varepsilon_h$. □

3 Power solutions and polarisation matrices

Operators (37), (26) are isomorphisms merely under conditions (27). When increasing (resp. decreasing) the weight index β in the first (resp. in the second) limit problem, it can be observed that the domain of the associated operator is growing, the kernel of the operator becomes nontrivial. Therefore, the operator looses the property of being an isomorphism and it turns out to be only an epimorphism. Elements of the kernels are involved into the subsequent asymptotic formulae. In this section we describe in details the form of such elements.

3.1 Power solutions of homegenuous system

The precision of the approximation of the true solution $u(\varepsilon, \cdot)$ by the asymptotic solution $\mathcal{U}(\varepsilon, \cdot)$ depends on our choice of integer J. How to select the appropriate value of J becomes clear after estimating the discrepancy in terms of J. At the moment, the value of $J \in \mathbb{N}$ is assumed to be fixed. For the linear space of vector–valued polynomials $p = (p_1, ..., p_T)^\top$ such that $\mathcal{L}(\nabla_x)p = 0$ in $\mathbb{R}^n$, $\mathbb{R}^n \ni x \mapsto p_j(x) \in \mathbb{R}$ and $\deg p_j \leq J$, we introduce the basis $\{U^1, ...U^N\}$ of homogeneous polynomials such that

$$U^h(zx) = z^{\tau_h} U^h(x) \quad \forall z \in \mathbb{R}, \quad \deg U^h = \tau_h \; ; \; \tau_h \leq \tau_{h+1} \,, \tag{70}$$

$$\sum_{j=1}^{T} U_j^k(\nabla_x) U_j^h(x)\Big|_{x=0} = \delta_{h,k} \; . \tag{71}$$

Let $\Phi = (\Phi^1, ..., \Phi^T)$ be the fundamental matrix of the operator $\mathcal{L}(\nabla_x)$ in $\mathbb{R}^n$, i.e., the columns $\Phi^1, ..., \Phi^T$ are given as the solutions to the systems of equations

$$\mathcal{L}(\nabla_x)\Phi^j(x) = e^j \delta(x), \quad x \in \mathbb{R}^n \; , \tag{72}$$

where $j = 1, ..., n$; $e^j = (\delta_{j,1}, ..., \delta_{j,T})^\top$ is the j–th vector of the canonical basis in $\mathbb{R}^n$, $\delta_{j,k}$ denotes the Kronecker symbol and $\delta \in \mathcal{D}'(\mathbb{R}^n)$ is the Dirac mass concentrated at $x = 0$.

It is well known, that for $n \geq 3$ the fundamental matrix satisfies the relation

$$\Phi^j(zx) = z^{2-n} \Phi^j(x) \quad \forall z \in \mathbb{R}_+ \; .$$

The power functions U^{-k}, homogeneous of degree $\deg U^{-k} = 2 - n - \tau_k$, are defined in terms of the columns Φ^j and the polynomials U^k

$$U^{-k}(x) = \sum_{j=1}^{T} U_j^k(-\nabla_x)\Phi^j(x) \; . \tag{73}$$

It is clear that the functions $U^{\pm 1}, ..., U^{\pm N}$, defined by (70) for $+k$ and by (73) for $-k$, are the power solutions of the homogeneous system in punctured space

$$\mathcal{L}(\nabla_x)U(x) = 0 \; , \quad x \in \mathbb{R}^n \setminus \mathcal{O} \; . \tag{74}$$

As it is shown in ([43], § 6.1, 6.4), [50], [52], these functions form a basis in the linear space of power solutions with the degree $\lambda \in \mathbb{C}$ such that the real part $\mathrm{Re}\lambda \in (1 - n - J, 1 + J)$.

3.2 Special solutions to second limit problem

The proof of the following Proposition 3.1 can be found in references ([43], § 6.1, 6.4), [50], [52].
Remark *In* [50], [52] *the results on the existence of solutions to elliptic boundary value problems are given in weighted Sobolev spaces. Such results can be established in the weighted Hölder spaces by applying the general method of* [31] *in the same way as in the proof of Theorem* 2.2.

Proposition 3.1 **1**) *The linear space of solutions to the homogeneous problem* (21)–(22) *in the space* $\Lambda^{l+1,\alpha}_{\gamma}(G)^T$ *with the weight index*

$$\gamma \in (l+\alpha-J, l+1+\alpha-J) \tag{75}$$

is given by the linear hull of the set of functions $\zeta^1,...,\zeta^N$, *of the form* $\zeta^j = U^j + z^j$, *where* $z^j \in \Lambda^{l+1,\alpha}_{\gamma}(G)^T$ *is the solution to problem* (21)–(22) *with the right hand sides* $f^\omega = 0$ *and* $g^\omega = -\mathcal{B}^\omega U^j$ *(Theorem* 2*). Furthermore, the functions* ζ^j *admit the representation*

$$\zeta^j(\xi) = U^j(\xi) + \sum_{h=1}^{N} m^\omega_{jh} U^{-h}(\xi) + \widetilde{\zeta}^j(\xi)\ , \tag{76}$$

where $\widetilde{\zeta}^j$ *belongs to the space* $\Lambda^{l+1,\alpha}_{\sigma}(G)^T$ *with the weight index*

$$\sigma \in (l-1+\alpha+J+n, l+\alpha+J+n)\ , \tag{77}$$

and the coefficients m^ω_{jh} *form a symmetric* $N\times N$ *polarisation matrix* m^ω.

2) *If* $\mathcal{B}^\omega$ *is the Dirichlet boundary operator, i.e.,* $\mathcal{B}^\omega w = w$, *then* m^ω *is a negative definite matrix.*

3) *If* $\mathcal{B}^\omega$ *is the Neumann boundary operator,* $\mathcal{B}^\omega = \mathcal{N}^\omega$, *then* m^ω *is a non–negative matrix which gives rise to a positive linear operator while restricted to the subspace* $\Pi^\omega \subset \mathbb{C}^N$. *The subspace* Π^ω *is defined as the orthogonal complement to the linear hull of the family of columns* $(b^p_1,...,b^p_N)^\top$ *consisting of the coefficients defined by the decomposition*

$$p(x) = \sum_{h=1}^{N} b^p_h U^h(x)\ . \tag{78}$$

of polynomials in $P_J(\omega) = \{p \in P(\omega)\ :\ \deg p \le J\}$ *(see* (16)*).*

3.3 Special solutions to the first limit problem

Let us consider first limit problem (17)–(18). As before, we replace (17)–(18) by (63)–(64) and denote by $\eta^1,...,\eta^N$ the *generalised* (cf. [64]) Green's functions, and their derivatives with respect to the second argument,

$$\mathcal{L}(\nabla_x)\eta^k(x) + \sum_{q=1}^{\ell} h^q(x)(\eta^k, h^q)_\Omega = U^k(-\nabla_x)\delta(x)\ ,\quad x\in\Omega\ , \tag{79}$$

$$\mathcal{B}^\Omega(x,\nabla_x)\eta^k(x) = 0\ ,\quad x\in\partial\Omega\ , \tag{80}$$

the poles of all functions η^j are located at the origin $x = \mathcal{O}$.
The following proposition is but the same as Theorem 2.9 in [50], the only difference is the presence of the non–local term in (79). We provide the proof for convenience of the reader.

Proposition 3.2 *The linear space of solutions to the homogeneous problem* (63)–(64) *in the space* $\Lambda_\sigma^{l+1,\alpha}(\Omega)^T$ *with weight index* (77) *is the linear hull of the functions* $\eta^1, ..., \eta^N$. *The solution* η^j *of* (79)–(80) *can be represented as the sum* $\eta^j(x) = U^j(x) + y^j(x)$, *where* $y^j \in \Lambda_\gamma^{l+1,\alpha}(\Omega)^T$ *denotes the solution to problem* (63)–(64) *with the right hand sides*

$$(81) \qquad f^j = -\sum_{p=1}^{\ell} h^p(U^{-j}, h^p)_\Omega \in C_0^\infty(\Omega \setminus \mathcal{O})^T \ , \quad g^j = -\mathcal{B}^\omega U^{-j} \in C^\infty(\partial\Omega)^T \ .$$

In addition, we have the representation

$$(82) \qquad \eta^j(x) = U^{-j}(x) + \sum_{h=1}^{N} m_{jh}^\Omega U^h(x) + \widetilde{\eta}^j(x) \ ,$$

where the remainder $\widetilde{\eta}^j$ *belongs to the space* $\Lambda_\gamma^{l+1,\alpha}(\Omega)$ *with weight index* (75), *the polarisation matrix* m^Ω *with the coefficients* m_{jk}^Ω *defined by formula* (82) *is a symmetric* $N \times N$ *matrix. The matrix* m^Ω *is positive definite in the case of the Dirichlet boundary conditions on* $\partial\Omega$.

Proof
It can be verified, by direct evaluations of the norms, taking into account (23)–(24), (77) and (27), that the function U^{-j} belongs to the space $\Lambda_\sigma^{l+1,\alpha}(\Omega)^T$ but does not belong to the space $\Lambda_\gamma^{l+1,\alpha}(\Omega)^T$. In particular, $U^{-j} + y^j \in \Lambda_\sigma^{l+1,\alpha}(\Omega)^T$, and $U^{-j} + y^j \notin \Lambda_\gamma^{l+1,\alpha}(\Omega)^T$. Furthermore, owing to (72), (73) and (81), the sum $\eta^j = U^{-j} + y^j$ solves the problem (79)–(80) with the Dirac mass δ in the right hand side. Let us point out that for any boundary value problem posed in weighted spaces, the specific problem can be defined properly only in the *punctured* domain $\Omega \setminus \mathcal{O}$, and in general it is not the case in the whole domain Ω. This is due to the fact that in the punctured domain the distributions supported at the origin $\mathcal{O}$ are ignored and actually, the function η^j is a solution to homogeneous problem (63)–(64) in the class $\Lambda_\sigma^{l+1,\alpha}(\Omega)^T$.

Since the set $\{U^{\pm 1}, ..., U^{\pm N}\}$ constitutes the basis in the linear space of power solutions of degree determined by the required precision of approximation, we regard the point $\mathcal{O}$ as the vertex of the complete cone $\mathbb{R}^n \setminus \mathcal{O}$. Therefore, we can apply the general theorem on asymptotic expansions of solutions to elliptic problems near conical points (see [22], [31], and also Theorems 3.5.6, 4.2.1 in [43]) to the solutions $\eta \in \Lambda_\sigma^{l+1,\alpha}(\Omega)$ and $y \in \Lambda_\beta^{l+1,\alpha}(\Omega)$ of problem (63)–(64) with the right hand sides $\{f, g\} = 0$ and $\{f, g\} \in \mathcal{R}_\gamma^{l,\alpha}\Lambda(\Omega, \partial\Omega)$, respectively. In this way we derive the expansions

$$(83) \qquad \eta = \sum_{j=1}^{N} b_j^- U^{-j} + \widehat{\eta}, \quad b_j^- \in \mathbb{C}, \quad \widehat{\eta} \in \Lambda_\beta^{l+1,\alpha}(\Omega)^T \ ,$$

$$(84) \qquad y = \sum_{j=1}^{N} b_j^+ U^j + \widehat{y}, \quad b_j^+ \in \mathbb{C}, \quad \widehat{y} \in \Lambda_\gamma^{l+1,\alpha}(\Omega)^T \ .$$

From (83), taking into account the uniqueness in the space $\Lambda_\beta^{l+1,\alpha}(\Omega)$ of the solution to (63)–(64), we can conclude that η is given by a linear combination of the functions $\eta^1, ..., \eta^N$. On the other hand, representation (82) follows from (84). Therefore, in order to complete the proof, we should establish the symmetry of the polarisation matrix m^Ω.

Let us note, that by the definition of the functions h^j and according to equation (79), it follows that $\mathcal{L}(\nabla_x)\eta^k(x) = U^{-k}(-\nabla_x)\delta(x)$ for $x \in \mathbb{B}_1$. Integration by parts in $\Omega \setminus \overline{\mathbb{B}}_\varrho$, where $\mathbb{B}_\varrho$

is a ball of radius $\varrho < 1$, leads to

$$
\begin{aligned}
(85)\qquad & (f^j,\eta^k)_\Omega + (g^j, \mathcal{T}^\Omega \eta^k)_{\partial\Omega} = (\mathcal{L}y^j + \sum_{p=1}^{\ell} h^p (y^j, h^p)_\Omega, \eta^k)_{\Omega\setminus\mathbb{B}_\varrho} + \\
& + \left(\mathcal{B}^\Omega y^j, \mathcal{T}^\Omega \eta^k\right)_{\partial\Omega} = (y^j, \underbrace{\mathcal{L}\eta^k + \sum_{p=1}^{\ell} h^p (y^j, h^p)_\Omega}_{=0})_{\Omega\setminus\mathbb{B}_\varrho} + \\
& +(\mathcal{T}^\Omega y^j, \underbrace{\mathcal{B}^\Omega \eta^k}_{=0})_{\partial\Omega} + \left(y^j, \mathcal{N}^{\mathbb{S}_\varrho}\eta^k\right)_{\mathbb{S}_\varrho} - \left(\mathcal{N}^{\mathbb{S}_\varrho} y^j, \eta^k\right)_{\mathbb{S}_\varrho},
\end{aligned}
$$

where $\mathbb{S}_\varrho$ is the sphere $\partial\mathbb{B}_\varrho$, $\mathcal{N}^{\mathbb{S}_\varrho}$ is the Neumann operator in the Green's formula (8) with Ω replaced by $\Omega \setminus \overline{\mathbb{B}}_\varrho$, and the normal vector on $\mathbb{S}_\varrho$ is directed inside the ball $\mathbb{B}_\varrho$. The left hand side of (85) is independent of ϱ. After neglecting terms $o(\varrho)$ the limit as $\varrho \to 0^+$ of the right hand side becomes

$$
\begin{aligned}
& \left(\sum_{h=1}^{N} m^\Omega_{jh} U^h, \mathcal{N}^{\mathbb{S}_\varrho}\left\{U^{-k} + \sum_{p=1}^{N} m^\Omega_{kp} U^p\right\}\right)_{\mathbb{S}_\varrho} - \left(\mathcal{N}^{\mathbb{S}_\varrho} \sum_{h=1}^{N} m^\Omega_{jh} U^h, U^{-k} + \sum_{p=1}^{N} m^\Omega_{kp} U^p\right)_{\mathbb{S}_\varrho} = \\
& = \left(\mathcal{L}\sum_{h=1}^{N} m^\Omega_{jh} U^h, U^{-k} + \sum_{p=1}^{N} m^\Omega_{kp} U^p\right)_{\mathbb{B}_\varrho} - \left(\sum_{h=1}^{N} m^\Omega_{jh} U^h, \mathcal{L}\left\{U^{-k} + \sum_{p=1}^{N} m^\Omega_{kp} U^p\right\}\right)_{\mathbb{B}_\varrho} = \\
(86)\qquad & = -\left(\sum_{h=1}^{N} m^\Omega_{jh} U^h, \mathcal{L}U^{-k}\right)_{\mathbb{B}_\varrho} = -\sum_{h=1}^{N} m^\Omega_{jh} U^h \int_{\mathbb{B}_\varrho} (\overline{U_k(-\nabla_x)\delta(x)})^\top U^k(x)dx \\
& = -\sum_{h=1}^{N} m^\Omega_{jh} U^k(\nabla_x)^\top U^h(x)\Big|_{x=0} = -\sum_{h=1}^{N} m^\Omega_{jh}\delta_{k,h} = -m^\Omega_{jk}\ .
\end{aligned}
$$

Remark

1) *The remainders $\widetilde{y}^j$ and $\widetilde{\eta}^k$ in the representations of y^j and η^k, respectively, give rise to integrals vanishing with $\varrho \to 0^+$. For example,*

$$
\begin{aligned}
& \left|\int_{\mathbb{S}_\varrho} \widetilde{\eta}^k(x)^\top \mathcal{N}^{\mathbb{S}_\varrho}(x,\nabla_x)\eta^k(x)ds_x\right| \le C\int_{\mathbb{S}_\varrho} \left|\widetilde{\eta}^k(x)\right| r^{2-n-J-1} ds_x \\
& \le \|\widetilde{\eta}^k; \Lambda^{l+1,\alpha}_\gamma(\Omega)\| \varrho^{-(\gamma-l-1-\alpha)} \varrho^{2-n-J-1}\varrho^{n-1} = C\varrho^{-\gamma+l+1+\alpha+J}\ ,
\end{aligned}
$$

where we take into account that the leading singularity of the function η^k is of the order r^{2-n-J}, and the exponent $-\gamma + l + 1 + \alpha + J > 0$ by (75).

2) *We have used the Green's formula inside the ball $\mathbb{B}_\varrho$, therefore, the sign of the Neumann operator $\mathcal{N}^{\mathbb{S}_\varrho}$ in the formula should be changed.*

3) *The Green's formula in general is applied in the sense of the theory of distributions.*

We transform the left hand side of (85) which equals to $-m^{\Omega}_{jk}$ by (86). Using (81) it follows that

$$m^{\Omega}_{jk} = -(f^j, \eta^k)_{\Omega} - (g^j, \mathcal{T}^{\Omega}\eta^k)_{\partial\Omega} = \tag{87}$$

$$= \sum_{p=1}^{\ell} (U^j, h^p)_{\Omega}(h^p, U^{-k})_{\Omega} + (\mathcal{B}^{\Omega}U^{-j}, \mathcal{T}^{\Omega}y^k)_{\partial\Omega} - \tag{88}$$

$$-\left\{ (\mathcal{L}y^j + \sum_{p=1}^{\ell} h^p (y^j, h^p)_{\Omega}, y^k)_{\Omega} + (\mathcal{B}^{\Omega}y^j, \mathcal{T}^{\Omega}y^k)_{\partial\Omega} \right\} .$$

We have to verify that $m^{\Omega}_{jk} = \overline{m^{\Omega}_{kj}}$. For the first term in the right hand side of (88) such a symmetry is straightforward. For the second term, first, the integration by parts over $\mathbb{R}^n \setminus \Omega$ is performed. Then, we observe that the integral over $\mathbb{R}^n \setminus \Omega$ converges since $|U^{-j}(x)| \le cr^{2-n}$, altogether it leads to

$$(\mathcal{B}^{\Omega}U^{-j}, \mathcal{T}^{\Omega}U^{-k})_{\partial\Omega} = (\mathcal{T}^{\Omega}U^{-j}, \mathcal{B}^{\Omega}U^{-k})_{\partial\Omega} + (\mathcal{L}U^{-j}, U^{-k})_{\mathbb{R}^n\setminus\Omega} -$$
$$-(U^{-j}, \mathcal{L}U^{-k})_{\mathbb{R}^n\setminus\Omega} = (\mathcal{T}^{\Omega}U^{-j}, \mathcal{B}^{\Omega}U^{-k})_{\partial\Omega} .$$

Finally, an application of the Green's formula in the interior of Ω yields

$$(\mathcal{L}y^j + \sum_{p=1}^{\ell} h^p (y^j, h^p)_{\Omega}, y^k)_{\Omega} + (\mathcal{B}^{\Omega}y^j, \mathcal{T}^{\Omega}y^k)_{\partial\Omega} =$$
$$= (y^j, \mathcal{L}y^k + \sum_{p=1}^{\ell} h^p (y^k, h^p)_{\Omega})_{\Omega} + (\mathcal{T}^{\Omega}y^j, \mathcal{B}^{\Omega}y^k)_{\partial\Omega} ,$$

which completes the proof of the symmetry of the polarisation matrix.

For the Dirichlet problem, $\mathcal{B}^{\Omega}u = u$ and $\mathcal{T}^{\Omega} = -\mathcal{N}^{\Omega}$, we have $\ell = 0$ and the sums with respect to p vanish in (88). Therefore, writing Green's formulae (8) in $\mathbb{R}^n \setminus \Omega$ for the functions U^{-j}, U^{-k} with the required change of the sign of the normal vector, and in Ω for the functions y^j, y^k, we obtain

$$m^{\Omega}_{jk} = -(U^{-j}, \mathcal{N}^{\Omega}U^{-k})_{\partial\Omega} + (y^j, \mathcal{N}^{\Omega}y^k)_{\partial\Omega} =$$
$$a(U^{-j}, U^{-k}; \mathbb{R}^n \setminus \Omega) + a(y^j, y^k; \Omega) =: Q^1_{jk} + Q^2_{jk} .$$

Each of the matrices Q^1 and Q^2 is non-negative definite (it enjoys the structure of the Gram matrix and the bilinear form satisfies $a(u, u; \Xi) \ge 0$). For example, if $\bar{c}^{\top}Q^1c = 0$ for some $c \in \mathbb{C}^N \setminus 0$, then

$$a\left(\sum_{j=1}^{N} c_j U^{-j}, \sum_{k=1}^{N} c_k U^{-k}; \mathbb{R}^n \setminus \Omega\right) = 0 ,$$

the equality means, according to (14), that $\sum_{j=1}^{N} c_j U^{-j}$ is a polynomial, which is impossible. Therefore, the matrix Q^1 as well as the matrix m^{Ω} are positive definite. □

4 Asymptotic expansions of solutions

It is natural to assume that functional (2) is invariant with respect to addition of polynomials from the kernel $P(\Omega)$ of boundary value problems (3)-(5) and (17)-(18) (see (15) and Theorem

2.3, Proposition 2.1)

$$\mathbb{J}_\varepsilon(u) = \mathbb{J}_\varepsilon(u+p) \quad \forall p \in P(\Omega)\ ,\ \varepsilon \in [0,1]\ . \tag{89}$$

Thus in (2) we can replace a solution to problem (3)–(5) by the solution to problem (46)–(48). In order to determine the first term of asymptotic expansion of $\mathbb{J}_\varepsilon(u)$ we are going to use the method of matched asymptotic expansions, in the framework developed in [50].

4.1 Asymptotic ansatzen

For simplicity, it is assumed that in (46)–(48) and (3)–(5) $g^\omega = 0$, $\{f^\Omega, g^\Omega\}$ does not depend on ε (see comments before Lemma 2.1), and

$$\{f^\Omega, g^\Omega\} \in \mathcal{R}^{l,\alpha}_\gamma \Lambda(\Omega, \partial\Omega)\,, \tag{90}$$

i.e., by (75) the right hand side f^Ω is defined in the whole domain Ω but decays, sufficiently fast, with $x \to \mathcal{O}$. We determine the first and the second expansions of the solution $u(\varepsilon, x)$ to problem (46)–(48) in the form

$$u(\varepsilon, x) \sim \mathcal{V}(\varepsilon, x) = v(x) + \sum_{j=1}^{N} a_j(\varepsilon)\eta^j(x)\,, \tag{91}$$

$$u(\varepsilon, x) \sim \mathcal{W}(\varepsilon, \varepsilon^{-1}x) = \sum_{j=1}^{N} b_j(\varepsilon)\zeta^j(\varepsilon^{-1}x)\,. \tag{92}$$

Here $v \in \Lambda^{l+1,\alpha}_\beta(\Omega)^T$ denotes the solution to problem (63)–(64) with right hand side (90), η^j and ζ^j the solutions defined in Propositions 8 and 7, respectively. The vectors in $\mathbb{C}^N$,

$$a(\varepsilon) = (a_1(\varepsilon), ..., a_N(\varepsilon))^\top\,, \quad b(\varepsilon) = (b_1(\varepsilon), ..., b_N(\varepsilon))^\top$$

are to be determined below. We introduce the rows of solutions (i.e., matrices of sizes $T \times N$)

$$\eta = (\eta^1, ..., \eta^N)\,, \quad \zeta = (\zeta^1, ..., \zeta^N)\,,$$

and rewrite (91)–(92) in the condensed form

$$\mathcal{V}(\varepsilon, x) = v(x) + \eta(x)a(\varepsilon)\,, \quad \mathcal{W}(\varepsilon, \xi) = \zeta(\xi)b(\varepsilon)\ .$$

For a function $y \in \Lambda^{l+1,\alpha}_\sigma(\Omega)^T$ which is represented as

$$y(x) = \sum_{j=1}^{N} d_j^+ U^{+j}(x) + \sum_{j=1}^{N} d_j^- U^{-j}(x) + \widetilde{y}(x)\,, \quad \widetilde{y} \in \Lambda^{l+1,\alpha}_\gamma(\Omega)^T\,,$$

we introduce the operators $\pi^\pm$ (projections) given by the columns of complex coefficients,

$$\pi^+ y = (d_1^+, ..., d_N^+)^T \in \mathbb{C}^N\,, \quad \pi^- y = (d_1^-, ..., d_N^-)^T \in \mathbb{C}^N\ .$$

The Taylor expansion of the solution v to problem (63)–(64) with right hand side (90) is defined in (83)–(84), namely

$$v(x) = \sum_{j=1}^{N} c_j U^j(x) + \widetilde{v}(x)\,, \quad \widetilde{v} \in \Lambda^{l+1,\alpha}_\gamma(\Omega)^T\,, \tag{93}$$

$$c = (c_1, ..., c_N)^\top = \pi^+ v \in \mathbb{C}^N\,, \quad \pi^- v = 0\,, \tag{94}$$

with the estimate

(95) $$\|\pi^+ v;\mathbb{C}^N\| + \|\widetilde{v};\Lambda^{l+1,\alpha}_\gamma(\Omega)^T\| \le c\|\{f^\Omega, g^\Omega\};\mathcal{R}^{l,\alpha}_\gamma\Lambda(\Omega,\partial\Omega)\| \ .$$

From the expansions (93) and (82), we find that

(96) $$\pi^+\mathcal{V} = c + m^\Omega a(\varepsilon)\,, \quad \pi^-\mathcal{V} = a(\varepsilon) \ .$$

Let us point out that according to (70) and (73),

(97) $$\varepsilon^{\tau_j} U^j(\xi) = U^j(x)\,, \quad \varepsilon^{2-n-\tau_j} U^{-j}(\xi) = U^{-j}(x) \ .$$

Therefore, we use the same notation

$$\pi^+ z = (d_1^+, ..., d_N^+)^\top\,, \quad \pi^- z = (d_1^-, ..., d_N^-)^\top$$

for any function $z \in \Lambda^{l+1,\alpha}_\gamma(G)^T$ of the form,

$$z(\xi) = \sum_{j=1}^N \varepsilon^{\tau_j} d_j^+ U^j(\xi) + \sum_{j=1}^N \varepsilon^{2-n-\tau_j} d_j^- U^{-j}(\xi) + \widetilde{z}(\xi)\,, \quad \widetilde{z} \in \Lambda^{l+1,\alpha}_\sigma(G)^T \ .$$

We introduce the diagonal matrix

(98) $$\mathcal{E} = \operatorname{diag}\{\varepsilon^{\tau_1}, ..., \varepsilon^{\tau_N}\} \ .$$

From (76) and (92) it follows that the coefficients of $\mathcal{W}$ satisfy the relations

(99) $$\pi^+\mathcal{W} = \mathcal{E}^{-1} b(\varepsilon)\,, \quad \pi^-\mathcal{W} = \varepsilon^{n-2}\mathcal{E} m^\omega b(\varepsilon)$$

which are used below in order to determine $b(\varepsilon)$.

4.2 The matching procedure

In general, an application of the method of matched asymptotic expansions necessitates the verification of the so–called *matching conditions*, we refer the reader e.g. to [73], [16], [37], [38], [25], for detailed description of the method. This means that the global approximate solution is constructed by combining the first and the second asymptotic expansions (91) and (92). As a result, an application of the matching conditions assures the admissible error of the approximation of the same magnitude as the required precision of the approximation. The error can occur in the intermediate region $\{r \sim \sqrt{\varepsilon}\}$, the region can be defined equivalently by the relation $\rho \sim \dfrac{1}{\sqrt{\varepsilon}}$. In our case it means that in the matching procedure we can restrict ourselves to the coefficients of polynomials $U^1, ..., U^N$ and of the derivatives $U^{-1}, ..., U^{-N}$ of the columns of fundamental solutions, and neglect the remainders $\widetilde{v}\,, \widetilde{\eta}^j\,, \widetilde{\zeta}^j$ in the expansions (93), (82) and (76), respectively. In view of the equalities (97) and taking into account the definition of the projections $\pi^\pm$, we can conclude that the required matching conditions imposed by the method, are equivalent in the particular case to the equalities

(100) $$\pi^+\mathcal{V} = \pi^+\mathcal{W}\,, \quad \pi^-\mathcal{V} = \pi^-\mathcal{W} \ .$$

The equalities ensure the approximate consistence of expansions (91) and (92) in the intermediate region. The equations (100), according to (96) and (99), take the form of the system of $2N$ linear algebraic equations for the unknown vectors $a(\varepsilon), b(\varepsilon) \in \mathbb{C}^N$, with given column $c = \pi^+ v$ defined in (94),

$$c + m^\Omega a(\varepsilon) = \mathcal{E}^{-1} b(\varepsilon)\,, \quad a(\varepsilon) = \varepsilon^{n-2} \mathcal{E} m^\omega b(\varepsilon)\ . \tag{101}$$

From (101) we readily derive

$$a(\varepsilon) = \varepsilon^{n-2} \mathcal{E} m^\omega \mathcal{E}(c + m^\Omega a(\varepsilon))\,,$$
$$b(\varepsilon) = \mathcal{E}(c + m^\Omega a(\varepsilon)) = \mathcal{E}(c + \varepsilon^{n-2} m^\Omega \mathcal{E} m^\omega b(\varepsilon))\,,$$

whence, with the $N \times N$ identity matrix $\mathcal{I}$, the unique solution to (101) takes the form

$$a(\varepsilon) = \{\mathcal{I} - \varepsilon^{n-2} \mathcal{E} m^\omega \mathcal{E} m^\Omega\}^{-1} \varepsilon^{n-2} \mathcal{E} m^\omega \mathcal{E} c\,, \tag{102}$$
$$b(\varepsilon) = \{\mathcal{I} - \varepsilon^{n-2} \mathcal{E} m^\Omega \mathcal{E} m^\omega\}^{-1} \mathcal{E} c\,, \tag{103}$$

Lemma 4.1 *The matrices $\mathcal{I} - \varepsilon^{n-2} \mathcal{E} m^\omega \mathcal{E} m^\Omega$ and $\mathcal{I} - \varepsilon^{n-2} \mathcal{E} m^\Omega \mathcal{E} m^\omega$ are invertible for small $\varepsilon > 0$. If the columns of coefficients of linear combinations* (91) *and* (92) *are defined by* (102) *and* (103)*, then the first and the second expansions of the solution $u(\varepsilon, x)$ to problem* (3)-(4) *are matched, i.e., the asymptotic terms detached in $\mathcal{V}$ and $\mathcal{W}$ according to decompositions* (93), (76) *and* (82) *coincide and relations* (100) *are verified.*

4.3 The global approximation

We choose the simplest construction of the global asymptotic approximation

$$\mathcal{U}(\varepsilon, x) = \mathcal{V}(\varepsilon, x) + \mathcal{W}(\varepsilon, x) - \{U^+(x)\pi^+\mathcal{V} + U^-(x)\pi^-\mathcal{V}\}\ , \tag{104}$$

where we denote by $U^\pm(x)$ the rows $(U^{\pm 1}(x), ..., U^{\pm N}(x))$ with the power solutions $U^{\pm j}(x)$, $j = 1, ..., N$.

In view of (100) the expression

$$U^+(x)\pi^+\mathcal{V} + U^-(x)\pi^-\mathcal{V} = U^+(x)\pi^+\mathcal{W} + U^-(x)\pi^-\mathcal{W}$$

is common asymptotic term in both approximations $\mathcal{V}(\varepsilon, x)$ and $\mathcal{W}(\varepsilon, \varepsilon^{-1}x)$ and that is the reason why it has been subtracted in (104).

Remark *The perturbation of domain Ω by the removal of a small cavity is of simple geometrical nature, and this implies the lack of cutoff functions in* (104)*. In the case of a perturbation of the domain near a conical point, the functions $\mathcal{V}$ and $\mathcal{W}$ may not be defined everywhere in $\Omega(\varepsilon)$ so the global approximation is determined as follows*

$$\begin{aligned}\mathcal{U}(\varepsilon, x) = (1 - \chi_1(\varepsilon, x))\mathcal{V}(\varepsilon, x) + \chi_0(x)\mathcal{W}(\varepsilon, \varepsilon^{-1}x) - \\ (1 - \chi_1(\varepsilon, x))\chi_0(x)\{U^+(x)\pi^+\mathcal{V} + U^-(x)\pi^-\mathcal{V}\}\,,\end{aligned} \tag{105}$$

where χ_1 and χ_0 are cutoff functions in (56) *Construction* (105)*, which takes form* (104) *if we put $1 - \chi_1 = 1$, $\chi_0 = 1$, is proposed for the first time in* [49]*, we refer also e.g. to (*[38]*; Chapter* 2*), (*[43]*;Chapter* 6*),* [50] *and many papers where the method is applied. We point out, that using formula* (105) *or* (104) *does not lead to any loose of asymptotic precision in contrast to the usage of the partition of unity $1 - \chi_{\frac{1}{2}}$ and $\chi_{\frac{1}{2}}$ (see* (56)*) and the global approximation*

$$(1 - \chi_{\frac{1}{2}}(\varepsilon, x))\mathcal{V}(\varepsilon, x) + \chi_{\frac{1}{2}}(\varepsilon, x)\mathcal{W}(\varepsilon, x) \tag{106}$$

(see [16], [14], [15] *and others). We emphasise that definition* (67) *of the approximate solution* $\mathcal{U}(\varepsilon, x)$ *is not founded on any complex construction, neither* (105) *nor* (67)*. It is possible since both auxiliary solutions* $v(\varepsilon, x)$ *and* $w(\varepsilon, \xi)$ *decay sufficiently fast with* $x \to \mathcal{O}$ *and* $\xi \to \infty$*, respectively. The same form of the global approximate solution is used in* (109) *below.*

Including to $\mathcal{W}(\varepsilon, x)$ or to $\mathcal{V}(\varepsilon, x)$ the terms subtracted in (104) we arrive at the expression for global approximation (104), respectively

$$\mathcal{U}(\varepsilon, x) = v(x) + \eta(x)a(\varepsilon) + \widetilde{\zeta}(\varepsilon^{-1}x)b(\varepsilon)\,, \tag{107}$$

$$\mathcal{U}(\varepsilon, x) = \widetilde{v}(x) + \widetilde{\eta}(x)a(\varepsilon) + \zeta(\varepsilon^{-1}x)b(\varepsilon)\,, \tag{108}$$

where the matrix functions $\widetilde{\eta}$ and $\widetilde{\zeta}$ stand for the rows $(\widetilde{\eta}_1, ..., \widetilde{\eta}_N)$ and $(\widetilde{\zeta}_1, ..., \widetilde{\zeta}_N)$ of remainders in the expansions (82) and (76), respectively. Formulae (107)–(108) follow from (104), taking into account the equalities (96)–(100), which leads in particular to

$$\begin{aligned} \mathcal{V}(\varepsilon, x) - U^+(x)\pi^+\mathcal{V} - U^-(x)\pi^-\mathcal{V} &= \widetilde{v}(x) + \widetilde{\eta}(x)a(\varepsilon) \\ \mathcal{W}(\varepsilon, x) - U^+(x)\pi^+\mathcal{W} - U^-(x)\pi^-\mathcal{W} &= \widetilde{\zeta}(\xi)b(\varepsilon)\ . \end{aligned}$$

On the other hand, in Proposition 11 below, we determine the third representation of the approximate solution $\mathcal{U}(\varepsilon, x)$ exactly in the form used for the approximation of shape functionals,

$$\mathcal{U}(\varepsilon, x) = v(x) + y(x)a(\varepsilon) + z(\varepsilon^{-1}x)b(\varepsilon)\,, \tag{109}$$

where, by definition, $y(x) = \eta(x) - U^-(x)$ and $z(\xi) = \zeta(\xi) - U^+(\xi)$.

Theorem 4.1 *If* $g^\omega = 0$ *and the data satisfy the condition* (90)*, then the norm of the difference between the solution* $u(\varepsilon, x)$ *to problem* (46)–(48) *and its asymptotic approximation* (104) *satisfies the estimate*

$$\|u - \mathcal{U}; \Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))\| \le c\varepsilon^{\min\{\beta-\gamma, \sigma-l-1-\alpha\}}\|\{f^\Omega, g^\Omega\}; \mathcal{R}_\gamma^{l,\alpha}\Lambda(\Omega, \partial\Omega)\|\,, \tag{110}$$

where β, γ, σ *are the weight indices defined by* (27), (75) *and* (77)*, respectively, the constant* c *depends on the choice of weight indices, while* c *is independent of* $\varepsilon \in (0, 1]$ *and of* $\{f^\Omega, g^\Omega\}$ *in* $\mathcal{R}_\gamma^{l,\alpha}(\Omega, \partial\Omega)$.

Proof We estimate the *discrepancies* of the approximate solution $\mathcal{U}$ in the boundary conditions (4)–(5) and in equation (3). Note that, e.g. the discrepancy of $\mathcal{U}$ in (4) implies, by definition, the difference $\mathcal{B}^\Omega u - \mathcal{B}^\Omega\mathcal{U} = g^\Omega - \mathcal{B}^\Omega\mathcal{U}$.
Using the representation (107)–(108) we obtain

$$\begin{aligned} \mathcal{B}^\Omega\mathcal{U} &= \mathcal{B}^\Omega v + \mathcal{B}^\Omega\eta a(\varepsilon) + \mathcal{B}^\Omega\widetilde{\zeta}b(\varepsilon) = g^\Omega + \mathcal{B}^\Omega\widetilde{\zeta}b(\varepsilon)\,, \\ \mathcal{B}^\omega\mathcal{U} &= \mathcal{B}^\omega\widetilde{v} + \mathcal{B}^\omega\widetilde{\eta}a(\varepsilon) + \mathcal{B}^\omega\zeta b(\varepsilon) = \mathcal{B}^\omega\widetilde{v} + \mathcal{B}^\omega\widetilde{\eta}a(\varepsilon) \end{aligned} \tag{111}$$

Thus, the discrepancy in boundary condition (4) satisfies

$$\begin{aligned} &\|\mathcal{B}^\Omega\widetilde{\zeta}(\varepsilon^{-1}\cdot); \prod_{j=1}^T \Lambda_\beta^{l+1-r_j^\Omega,\alpha}(\partial\Omega)\| \\ &= \sum_{j=1}^T \varepsilon^{\beta-l-1+r_j^\Omega-\alpha}\|\varepsilon^{-r_j^\Omega}\mathcal{B}_j^\Omega(\varepsilon\cdot, \nabla_\xi)\widetilde{\zeta}; \Lambda_\beta^{l+1-r_j^\Omega,\alpha}(\partial\Omega_{\frac{1}{\varepsilon}})\| \le \\ &\le c\varepsilon^{\beta-l-1-\alpha}\|\widetilde{\zeta}; \Lambda_\beta^{l+1,\alpha}(\partial\Omega_{\frac{1}{\varepsilon}} \setminus \mathbb{B}_{\frac{1}{\varepsilon}})\| \le c\varepsilon^{\beta-l-1-\alpha}\varepsilon^{\sigma-\beta}\|\widetilde{\zeta}; \Lambda_\sigma^{l+1,\alpha}(G)\| \\ &\le c_\zeta\varepsilon^{\sigma-l-1-\alpha}\,. \end{aligned} \tag{112}$$

Similarly, two terms of the discrepancy in boundary conditions in (5) are estimated by

$$\|\mathcal{B}^\omega \widetilde{v}; \prod_{j=1}^{T} \Lambda_\beta^{l+1-r_j^\omega,\alpha}(\partial\omega_\varepsilon)\| \leq \tag{113}$$

$$\leq c\|\widetilde{v}; \Lambda_\beta^{l+1,\alpha}(\mathbb{B}_\varepsilon)\| \leq c\varepsilon^{\beta-\gamma}\|\widetilde{v}; \Lambda_\gamma^{l+1,\alpha}(\mathbb{B}_\varepsilon)\| \leq c\varepsilon^{\beta-\gamma}\|\widetilde{v}; \Lambda_\gamma^{l+1,\alpha}(\Omega)\| \leq c\mathbf{N}\,,$$

and

$$\|\mathcal{B}^\omega \widetilde{\eta}; \prod_{j=1}^{T} \Lambda_\beta^{l+1-r_j^\omega,\alpha}(\partial\omega_\varepsilon)\| \leq \tag{114}$$

$$\leq c\|\widetilde{\eta}; \Lambda_\beta^{l+1,\alpha}(\mathbb{B}_\varepsilon)\| \leq c\varepsilon^{\beta-\gamma}\|\widetilde{\eta}; \Lambda_\gamma^{l+1,\alpha}(\mathbb{B}_\varepsilon)\| \leq c_\eta \varepsilon^{\beta-\gamma}\,,$$

where $\mathbf{N}$ denotes the norm of the right hand side in (90).
By (102)–(103) and (95)

$$\varepsilon^{2-n}\|a(\varepsilon); \mathbb{C}^N\| + \|b(\varepsilon); \mathbb{C}^N\| \leq c\|\pi^+ v; \mathbb{C}^N\| \leq c\mathbf{N}\,. \tag{115}$$

From (102)–(103) and (112)–(114) it follows now that

$$\|\{0, \mathcal{B}^\Omega \mathcal{U} - g^\Omega, \mathcal{B}^\omega \mathcal{U}\}; \mathcal{R}_\beta^{l,\varepsilon}(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon)\| \leq \tag{116}$$

$$\leq c\varepsilon^{\min\{\beta-\gamma, \sigma-l-1-\alpha\}}\mathbf{N}\,.$$

We transform system (3) using the representation (107),

$$\mathcal{L}\mathcal{U} + \sum_{p=1}^{d} h^p(\mathcal{U}, h^p)_\Omega = \mathcal{L}v + \sum_{p=1}^{d} h^p(v, h^p)_\Omega +$$

$$\left\{\mathcal{L}\eta + \sum_{p=1}^{d} h^p(\eta, h^p)_\Omega\right\} a(\varepsilon) + \mathcal{L}\widetilde{\zeta} b(\varepsilon) + \sum_{p=1}^{d} h^p(\widetilde{\zeta}, h^p)_{\Omega\setminus\mathbb{B}_1} b(\varepsilon) =$$

$$= f^\Omega + 0 + 0 + 0 + \sum_{p=1}^{d} h^p(\widetilde{\zeta} b(\varepsilon), h^p)_{\Omega\setminus\mathbb{B}_1}\,.$$

The scalar products are bounded,

$$|(\widetilde{\zeta}, h^p)_{\Omega\setminus\mathbb{B}_1}| \leq c\|\widetilde{\zeta}; \Lambda_\sigma^{l+1,\alpha}(G)\| \int_1^{\operatorname{diam}\Omega} \left(\frac{r}{\varepsilon}\right)^{l+1+\alpha-\sigma} r^{n-1} dr \leq$$

$$c_\zeta \varepsilon^{\sigma-l-1-\alpha}\,,$$

and we can estimate the discrepancy in system (3) as follows

$$\|\mathcal{L}\mathcal{U} + \sum_{p=1}^{d} h^p(\mathcal{U}, h^p)_\Omega - f^\Omega; \Lambda_\beta^{l-1,\alpha}(\Omega(\varepsilon))\| \leq c\varepsilon^{\sigma-l-1-\alpha}\mathbf{N}\,. \tag{117}$$

To complete the proof it is sufficient to apply the uniform estimate of the norm of the inverse operator $(\mathcal{A}^\varepsilon)^{-1}$ (Theorem 2.4),

$$\|\mathcal{U} - u; \Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))\| \leq c\|\mathcal{A}_h^\varepsilon \mathcal{U} - \mathcal{A}_h^\varepsilon u; \mathcal{R}_\beta^{l,\alpha}\Lambda(\Omega(\varepsilon), \partial\Omega, \partial\omega_\varepsilon)\|\,, \tag{118}$$

and establish an upper bound of the right hand side in (118), using (117) and (116). □

4.4 Asymptotic formulae of required precision

The asymptotic approximation $\mathcal{U}$ of the solution u, depends on the integer J which determines the precision of the approximation. Our goal is to determine the first term of the asymptotic expansion of the associated integral functionals by an application of the Taylor formula, which can be quite involved (we refer to the Appendix for details), with the solution u replaced by the approximate solution $\mathcal{U}$. Such a procedure requires the appropriate precision of the asymptotic solution $\mathcal{U}$. We address the question of the appropriate choice of the integer J.

Since the restrictions (27), (75), (78) read as follows

$$0 < \beta - l - 1 - \alpha < n - 2\,,$$
$$-J - 1 < \gamma - l - 1 - \alpha < -J\,,$$
$$J + n - 2 < \sigma - l - 1 - \alpha < J + n - 1\,,$$

we find that

$$J < \beta - \gamma < J + n - 1\,,$$

which means that the exponent of ε in the upper bound from the formula (110) is located in the interval

$$(J, J + n - 1)\ . \tag{119}$$

For example, with a small δ, $0 < \delta < \frac{1}{2}$, and assuming the equalities

$$\beta - (l + 1 + \alpha) = n - 2 - \delta\,, \tag{120}$$
$$\gamma - (l + 1 + \alpha) = -J - 2\delta\,, \tag{121}$$
$$\sigma - (l + 1 + \alpha) = J + n + 2 + \delta \tag{122}$$

the upper bound in (110) takes the form

$$c\varepsilon^{J+n-2+\delta}\|\{f^\Omega, g^\Omega\}; \mathcal{R}^{l,\alpha}_\gamma \Lambda(\Omega, \partial\Omega)\| =: c\varepsilon^{J+n-2+\delta}\mathbf{N}_J\ . \tag{123}$$

It is clear now, how to select the integer J in order to assure the required precision of approximation of the functional (2) while replacing the true solution $u(\varepsilon, x)$ by the asymptotic solution $\mathcal{U}(\varepsilon, x)$.

Proposition 4.1 *Assume that the weight indices are given by* (120)–(122) *and*

$$q \le \min\{l + 1, J\}\ .$$

1) *If for any* $\varepsilon \in (0, \varepsilon_0]$ *and* $x \in \Omega(\varepsilon)$ *the following estimates hold*

$$|\nabla^p_x \mathcal{U}(\varepsilon, x)| \le c_p \mathbf{N}_J\,, \quad p = 0, 1, ..., q\,, \tag{124}$$

then for $\varepsilon \in (0, \varepsilon_0]$ *and* $x \in \Omega(\varepsilon)$*,*

$$|\nabla^p_x u(\varepsilon, x)| \le C_p \mathbf{N}_J\,, \quad p = 0, 1, ..., q\ . \tag{125}$$

2) *If the coefficients in* (91)–(92) *satisfy the inequalities*

$$|a_j(\varepsilon)| \le c\mathbf{N}_J\,, \quad |b_j(\varepsilon)| \le c\varepsilon^q \mathbf{N}_J\,, \quad j = 1, ..., N\,, \tag{126}$$

then the inequalities (124) *and* (125) *are obtained with the constant* $\mathbf{N}_J$ *defined by* (123).

Proof
1) Since $r > c\varepsilon$ in $\Omega(\varepsilon)$, taking into account (120), (110), and (123), it follows that for $x \in \Omega(\varepsilon)$

$$|\nabla_x^p u(\varepsilon,x) - \nabla_x^p \mathcal{U}(\varepsilon,x)| \le c\varepsilon^{l+1+\alpha-\beta-p} r^{\beta-l-1-\alpha+p} |\nabla_x^p u(\varepsilon,x) - \nabla_x^p \mathcal{U}(\varepsilon,x)|$$
$$\le c\varepsilon^{2-n+\delta-p} \|u - \mathcal{U}; \Lambda_\beta^{l+1,\alpha}(\Omega(\varepsilon))\| \le c\varepsilon^{J-p}\mathbf{N}_J\ .$$

2) We use representation (109) of the approximate solution (104), which can be obtained directly from (104) and (100). To complete the proof let us observe that

$$\begin{aligned} &|\nabla_x^p v(x)| \le c_p(|\pi^+ v| + \|\widetilde{v}; \Lambda_\gamma^{l+1,\alpha}(\Omega)\|)\,, \\ &|\nabla_x^p y(x)| \le c_p\,, \quad |\nabla_\xi^p z(\xi)| \le c_p |\xi|^{2-n-p}\,, \end{aligned} \tag{127}$$

hence, we can differentiate (109) taking into account that $\nabla_x^p = \varepsilon^{-p}\nabla_\xi^p$. Therefore, using (126), it is easy to see that (124) and (125) follow. □

The equalities (102)–(103) can be rewritten in the form

$$a(\varepsilon) = \varepsilon^{n-2}\mathcal{E}m^\omega\{\mathcal{I} - \varepsilon^{n-2}\mathcal{E}m^\Omega\mathcal{E}m^\omega\}^{-1}\mathcal{E}c\,, \tag{128}$$
$$b(\varepsilon) = \mathcal{E}c - \mathcal{E}m^\Omega\{\mathcal{I} - \varepsilon^{n-2}\mathcal{E}m^\Omega\mathcal{E}m^\omega\}^{-1}\mathcal{E}m^\omega\mathcal{E}c\,, \tag{129}$$

whence

$$|a_j(\varepsilon)| \le c\varepsilon^{n-2+\tau_j}|\pi^+ v|\,, \quad |b_j(\varepsilon)| \le c\varepsilon^{\tau_j}|\pi^+ v|\,, \tag{130}$$

which means that conditions (126) are satisfied for $q = 0$ and we have the following corollary.

Corollary 4.1 *Under our assumptions, for any J,*

$$|u(\varepsilon,x)| + |\mathcal{U}(\varepsilon,x)| \le C\mathbf{N}_J\,, \tag{131}$$
$$|\nabla_x u(\varepsilon,x)| + |\nabla_x \mathcal{U}(\varepsilon,x)| \le C\varepsilon^{-1}\mathbf{N}_J\ , \tag{132}$$

where the constant $\mathbf{N}_J$ is defined by (123).

Note that inequality (132) follows from (127).

5 Asymptotic expansions of shape functionals

Now we are in position to provide the first terms of asymptotic expansions of shape functionals. The asymptotics of the functional (2) are derived, by replacing the exact solution u and its gradient $\nabla_x u$ by its approximations $\mathcal{U}$ and $\nabla_x\mathcal{U}$, respectively, and by subsequent evaluation of the asymptotics of the resulting functional

$$\mathbb{J}_\varepsilon(\mathcal{U}) = \int_{\Xi(\varepsilon)} \mathcal{F}(x, \mathcal{U}(\varepsilon,x), \nabla_x\mathcal{U}(\varepsilon,x))ds_x\ . \tag{133}$$

Such an approach is applicable in view of formulae (131)–(132) and (101), however it requires additional explanations due to the presence of the factor ε^{-1} in the right hand side of (132).

Since the leading term of asymptotics for solution $u(\varepsilon,x)$ substantially depends on the type of boundary conditions on the interior boundary $\partial\omega_\varepsilon$, two different cases are considered, the second concerns the Neumann boundary conditions $\mathcal{B}^\omega = \mathcal{N}^\omega$ for which the upper bound in (132) changes for $C\mathbf{N}_J$. We point out that for any other boundary conditions, the Dirichlet or mixed conditions, on $\partial\omega_\varepsilon$, estimate (132) cannot be improved due to the growth of the gradients $\nabla_x u(\varepsilon, x)$ and $\nabla_x \mathcal{U}(\varepsilon, x)$, therefore we restrict our considerations to the functional (13), namely, to the resulting functional

$$\mathbb{J}_\varepsilon(\mathcal{U}) = \int_{\Xi(\varepsilon)} \mathcal{F}(x, \mathcal{U}(\varepsilon, x)) ds_x \ , \tag{134}$$

with the integrand depending exclusively on the asymptotic solution $\mathcal{U}$.

5.1 Boundary conditions different from Neumann conditions on $\partial\omega_\varepsilon$

In this section we deal with either the Dirichlet boundary conditions or the mixed boundary conditions on $\partial\omega_\varepsilon$. The Neumann conditions are considered in the next subsection.

We suppose that the linear space $P(\omega)$ from (16) excludes the constant columns

$$U^1, ..., U^\varkappa \ , \tag{135}$$

but includes the columns $U^{\varkappa+1}, ..., U^T$, where for $\varkappa = T$ the Dirichlet condition is prescribed on $\partial\omega_\varepsilon$, the case $\varkappa = 0$ leads to to the Neumann conditions which are not allowed in the section, so $T \geq \varkappa > 0$ and condition (71) is satisfied (without differential operators for constant columns), i.e.,

$$(U^j)^\top U^k = \delta_{j,k} \ , \quad j, k = 1, ..., \varkappa \ . \tag{136}$$

Let us set $l = 1, J = 0$ (i.e., the minimal admissible) and for $\delta \in (0, \frac{1}{2})$ we introduce, according to (120)–(122),

$$\beta = \alpha + n - \delta \ , \quad \gamma = 2 + \alpha - 2\delta \ , \quad \sigma = \alpha + n + \delta \ . \tag{137}$$

For such a choice of indices, conditions (27), (75), and (77) are satisfied. The upper bound in (110) takes the form

$$c\varepsilon^{n-2+\delta} \|\{f^\Omega, g^\Omega\}; \mathcal{R}^{1,\alpha}_\gamma \Lambda(\Omega, \partial\Omega)\| := c\varepsilon^{n-2+\delta} \mathbf{N}_0 \ . \tag{138}$$

Since, owing to the choice $J = 0$, only the constant vectors $U^1, ..., U^T$, are taken into account, it follows that $N = T$ and m^ω is a $T \times T$ matrix. The matrix $(-m^\omega)$ is the so-called *capacity* matrix in the case of the Dirichlet boundary conditions on $\partial\omega$ and it becomes positive definite. Furthermore,

$$\mathcal{E} = \mathcal{I}_{T\times T} \ , \tag{139}$$

$$\pi^+ v = c = (c_1, ..., c_T)^\top = v(0) \ , \tag{140}$$

$$|a(\varepsilon) - \varepsilon^{n-2} m^\omega v(0)| \leq c\varepsilon^{2(n-2)} |v(0)| \ , \tag{141}$$

$$|b(\varepsilon) - v(0)| \leq c\varepsilon^{n-2} |v(0)| \ . \tag{142}$$

Note that (140) becomes true, provided we have $U^j = (\delta_{j,1}, ..., \delta_{j,T})^T$, which can be obtained after appropriate choice of the basis.

By Corollary 4.1 and Theorem 4.1,

$$|\mathbb{J}_\varepsilon(u) - \mathbb{J}_\varepsilon(\mathcal{U})| \leq C_\mathcal{F}(\max|u|, \max|\mathcal{U}|) \int_{\Xi(\varepsilon)} |u(\varepsilon, x) - \mathcal{U}(\varepsilon, x)| ds_x \leq$$

$$\leq C(\mathbf{N}_0) \|u - \mathcal{U}; \Lambda^{2,\alpha}_\beta(\Omega(\varepsilon))\| \int_{\Xi(\varepsilon)} r^{2+\alpha-\beta} ds_x \leq C(\mathbf{N}_0) \varepsilon^{n-2+\delta} \ .$$

Here, we use the convergence, for $d = n$ and $d = n-1$, of integrals

$$\int_{\Xi} r^{2+\alpha-\beta} = \int_{\Xi} r^{2-n+\delta} \,.$$

The required precision $O(\varepsilon^{n-2})$ of the approximation is bounded by $O(\varepsilon^d)$, therefore, employing representation (107) the asymptotics of the integral

$$\mathbb{J}_\varepsilon(\mathcal{U}) = \int_{\Xi(\varepsilon)} \mathcal{F}(x, v(x) + \eta(x)a(\varepsilon) + \widetilde{\zeta}(\varepsilon^{-1}x)b(\varepsilon))ds_x$$

can be determined by direct evaluations

$$\begin{aligned}
(143)\qquad & |\mathbb{J}_\varepsilon(\mathcal{U}) - \mathbb{J}_0(v) - \varepsilon^{n-2}\int_{\Xi} \mathcal{F}'_u(x, v(x))^\top \eta(x) ds_x m^\omega v(0)| \leq \\
& \leq C(\mathbf{N}_0)\left\{ \int_{\Xi(\varepsilon)} |\widetilde{\zeta}(\varepsilon^{-1}x)b(\varepsilon)| ds_x + \int_{\Xi(\varepsilon)} |\eta(x)(a(\varepsilon) - \varepsilon^{n-2}m^\omega v(0))| ds_x + \right. \\
& \left. + \int_{\Xi\setminus\Xi(\varepsilon)} |\mathcal{F}(x, v(x))| ds_x + \varepsilon^{n-2}\int_{\Xi\setminus\Xi(\varepsilon)} |\mathcal{F}'_u(x, v(x))^\top \eta(x)| ds_x |v(0)| \right\} =: \\
& =: C(\mathbf{N}_0)\{I^1 + I^2 + I^3 + I^4\}\,,
\end{aligned}$$

where $\mathcal{F}'_u(x, u)$ stands for the column of height T with the components $\frac{\partial}{\partial u_i}\mathcal{F}(x, u)$. The right hand side of (143) can be estimated by using formulae (76), (77) for I_1 and (82) for I_2, I_3. As a result we obtain

$$\begin{aligned}
|I^1| &\leq c|v(0)| \int_{\Xi(\varepsilon)} \left(\frac{\varepsilon}{r}\right)^{n-1} ds_x \leq c\mathbf{N}_0 \varepsilon^{n-1}(1 + \delta_{d,n-1}|\ln\varepsilon|)\,, \\
|I^2| &\leq c|v(0)|\varepsilon^{2(n-2)} \int_{\Xi(\varepsilon)} r^{2-n} ds_x \leq c\mathbf{N}_0 \varepsilon^{2(n-2)}\,, \\
|I^3| &\leq C_{\mathcal{F}}(\mathbf{N}_0)\mathrm{meas}_d(\Xi \setminus \Xi(\varepsilon)) \leq C_{\mathcal{F}}(\mathbf{N}_0)\varepsilon^d\,, \quad (\text{ note that } d \geq n-1) \\
|I^4| &\leq C'_{\mathcal{F}}(\mathbf{N}_0)\varepsilon^{n-2}\int_0^c r^{2-n} ds_x \leq c\mathbf{N}_0 \varepsilon^{n-2+2-n+d}\,,
\end{aligned}$$

and $|\ln\varepsilon|$ is present in the first line only for $d = n-1$. In derivation of the upper bounds for $|I^1|$ and $|I^2|$, inequalities (141)–(142) are used.

We are in position to establish the asymptotic formula valid for functional (13), i.e., with the integrand independent of the gradient $\nabla_x u$, and for all boundary conditions on $\partial\Omega$ and $\partial\omega_\varepsilon$, constructed according to (10) and (11). Unfortunately, such result is useless for the Neumann conditions on $\partial\omega_\varepsilon$ which generates null $T \times T$ polarisation matrix m^ω, we refer the reader to subsequent section for this case. The first statement of Theorem 5.1 concerns both cases of $d = n$ and $d = n-1$, but the second statement with the appropriate definition of the adjoint state [66], [70] is limited only to the case $\Xi = \Omega$. Otherwise, a minimal regularity with $l = 1$ of the right hand sides is required and at the very end we can forget about the weighted Hölder classes. The latter is possible in view of the embedding $\mathcal{R}^{1,\alpha}C(\Omega, \partial\Omega) \subset \mathcal{R}^{1,\alpha}_\gamma\Lambda(\Omega, \partial\Omega)$, which follows from the second condition in (137) combined with the fact that the powers of r in the norm $\|f^\Omega; \Lambda^{0,\alpha}_\gamma(\Omega)\|$ (see (25)) are positive. Finally, taking into account the properties (89) of functional (13), the compability conditions are imposed in order to assure the solvability of problem (17)-(18) thus we can dispose the nonlocal terms of problem (63)-(64).

To ensure the existence of a solution to problem (3)-(5) with $g^\omega = 0$, i.e. the fulfillement of compability condition (39), a correction $\widehat{g}$ is added to compensate the diminution by $(f^\Omega, p)_{\omega_\varepsilon}$ of the first term in (20) owing to the appearance of the cavity ω_ε and the passage from problem (17)-(18) to problem (3)-(5). Since

$$\left|(f^\Omega, p)_{\omega_\varepsilon}\right| \le C\varepsilon^n \|f^\Omega; C^{0,\alpha}(\Omega)\| \ ,$$

the correction $\widehat{g}$ can be taken so small

$$\|\widehat{g}; C^{1,\alpha}(\partial\Omega)\| \le C\varepsilon^n \|f^\Omega; C^{0,\alpha}(\Omega)\| \ , \tag{144}$$

that can be ignored when constructing the asymptotic solution $\mathcal{U}$.

Theorem 5.1 **1**) *Let* $\{f^\Omega, g^\Omega\} \in \mathcal{R}^{1,\alpha}C(\Omega, \partial\Omega)$ *and* $\{f, g\} = \{f^\Omega, g^\Omega + \widehat{g}\} \in \mathcal{R}^{1,\alpha}C(\Omega(\varepsilon), \partial\Omega(\varepsilon))$ *satisfy compability conditions* (20) *and* (39), *respectively, where* $\widehat{g}$ *fulfils* (144). *Then for functional* (13), *where* $u \in C^{2,\alpha}(\Omega(\varepsilon))^T$ *is a solution to problem* (3)–(5) *with* $g^\omega = 0$, *the following inequality holds,*

$$|\mathbb{J}_\varepsilon(u) - \mathbb{J}_0(v) - \varepsilon^{n-2} \int_\Xi \mathcal{F}_u'(x, v(x))^\top \eta(x) m^\omega v(0) ds_x| \le C^\delta_{f,g} \varepsilon^{n-2+\delta} \ . \tag{145}$$

The constant $C^\delta_{f,g}$ *depends on the norm* $\|\{f^\Omega, g^\Omega\}; \mathcal{R}^{1,\alpha}\Lambda(\Omega, \partial\Omega)\|$ *and* $\delta \in (0, 1/2)$, $v \in C^{2,\alpha}(\Omega)^T$ *is a solution to problem* (17)–(18), *any solution is admissible in the case of nontrivial kernel* $P(\Omega)$ *(see Theorem* 2.1*),* $\eta = (\eta^1, ..., \eta^T)$ *denotes the Green's matrix for boundary value problem* (63)–(64), *and* m^ω *is the polarisation matrix of size* $T \times T$.

By (135) *the left upper* $\varkappa \times \varkappa$ *blocs* $\left(m^\omega_{jk}\right)^{\varkappa}_{j,k=1}$ *of the polarisation matrix* m^ω *is non–trivial. If the Dirichlet condition is prescribed on* $\partial\omega_\varepsilon$, *then* $(-m^\omega)$ *is the positive definite capacity matrix.*
2) *In the case* $\Xi = \Omega$, *formula* (145) *turns into*

$$|\mathbb{J}_\varepsilon(u) - \mathbb{J}_0(v) - \varepsilon^{n-2} V(0)^\top m^\omega v(0)| \le C^\delta_{f,g} \varepsilon^{n-2+\delta} \ , \tag{146}$$

where by $V \in C^{2,\alpha}(\Omega)^T$ *is denoted a solution to the problem*

$$\mathcal{L}(\nabla_x) V(x) = \mathcal{F}_u'(x, v(x)) \ , \quad x \in \Omega \ , \tag{147}$$

$$\mathcal{B}^\Omega(x, \nabla_x) V(x) = 0 \ , \quad x \in \partial\Omega \ . \tag{148}$$

Proof To complete the proof it is sufficient to show that (146) is valid. The required formula is obtained from (145), taking into account the equality

$$V(0) = \int_\Xi \mathcal{F}_u'(x, v(x)) \eta(x) dx$$

which, in turn, implies the basic property of the Green's matrix η.

We point out that compability conditions for problem (147)-(148),

$$(\mathcal{F}_u', p)_\Omega = 0 \quad \forall p \in P(\Omega) \ ,$$

follows from property (89) by differentiation with respect to t of the equality

$$\int_\Omega \mathcal{F}(x, v(x) + tp(x)) dx = \int_\Omega \mathcal{F}(x, v(x)) dx \ .$$

Finally, the term $V(0)^\top m^\omega v(0)$ is independent on the choice of the solutions V and v, since by assumption (41) the polynomial $p \in P(\Omega) \subset P(\omega)$ verifies the condition $\mathcal{B}^\omega p = 0$ on $\partial\omega$, thus $m^\omega p(0) = 0$. □

Remark *The generalised Green's functions η^j, i.e., the solutions of problem* (63)-(64) *with $g^1 = 0$ and $f^1(x) = e^j\delta(x)$ (cf.* (72)*) are used in formula* (145)*. The existence of all Green's functions for problem* (17)-(18) *is not assured in the case of nontrivial kernel $P(\Omega)$. However the linear combinations $\eta(x)b$ are always defined for all columns $b \in \mathbb{R}^T$ which are orthogonal to the columns $p(\mathcal{O})$ with $p \in P(\Omega)$ (cf. compability condition* (20)*). The specific property of matrix m^ω which is indicated at the end in the proof of Theorem* 5.1 *makes it possible to evaluate the integral in* (145) *even in the case of nontrivial kernel $P(\Omega)$.*

5.2 Neumann boundary conditions on $\partial\omega_\varepsilon$

For $\mathcal{B}^\omega = \mathcal{N}^\omega$ the linear space $P(\omega)$ defined in (16) contains the constant vectors which means that the polarisation matrix is of the special structure with zeros in the first T rows and columns. This implies that the integral with factor ε^{n-2} in the formula (145) vanishes and the formula (145) does not furnish any useful information. Let us suppose that there exists at least one linear function $p_1(x)$ such that $\mathcal{N}^\omega p_1 \neq 0$, i.e., the submatrix $\left(m^\omega_{jk}\right)_{j,k=1}^{T(n+1)}$ of the polarisation matrix is non trivial. We consider general functional (2) which has the intrinsic property (89) and therefore, we can consider problem (46)–(48) instead of problem (3)–(5), with $g^\omega = 0$.
Let us set $l = 1$, $J = 2$, and define for $\delta \in (0, 1/2)$,

$$\beta = \alpha + n - \delta\,, \quad \gamma = \alpha - 2\delta\,, \quad \sigma = 2 + \alpha + n + \delta\,, \tag{149}$$

so it is easy to see that conditions (27), (75), and (77) are satisfied. The vector $c = \pi^+ v \in \mathbb{C}^N$ contains N coefficients of the Taylor expansion at the origin of the solution v to (63)–(64) with the right hand side

$$\{f^\Omega, g^\Omega\} \in \mathcal{R}^{1,\alpha}_\gamma \Lambda(\Omega, \partial\Omega)\,, \tag{150}$$

where N is decomposed in the following way, taking into account the degrees of polynomials,

$$N = \underset{\substack{\uparrow \\ \textit{constants}}}{T} + \underset{\substack{\uparrow \\ \textit{linear}}}{T\,n} + \underset{\substack{\uparrow \\ \textit{quadratic}}}{T(n^2-1)}$$

which results in the following decomposition of c,

$$c = \begin{pmatrix} c_{(0)} \\ c_{(1)} \\ c_{(2)} \end{pmatrix} = \pi^+ v = \begin{pmatrix} \pi^+_{(0)} v \\ \pi^+_{(1)} v \\ \pi^+_{(2)} v \end{pmatrix} \tag{151}$$

and we denote

$$\pi^+_\bullet v = \begin{pmatrix} \pi^+_{(1)} v \\ \pi^+_{(2)} v \end{pmatrix}$$

Furthermore,

$$\mathcal{E} = \mathrm{diag}\{\underbrace{1, ..., 1}_{T}, \underbrace{\varepsilon, ..., \varepsilon}_{Tn}, \underbrace{\varepsilon^2, ..., \varepsilon^2}_{T(n^2-1)}\} = \mathrm{diag}\{\mathcal{E}_{(0)}, \mathcal{E}_{(1)}, \mathcal{E}_{(2)}\}$$

and the polarisation matrix is decomposed into blocks as follows

$$m^\omega = \begin{pmatrix} m^\omega_{(00)} & m^\omega_{(01)} & m^\omega_{(02)} \\ m^\omega_{(10)} & m^\omega_{(11)} & m^\omega_{(12)} \\ m^\omega_{(20)} & m^\omega_{(21)} & m^\omega_{(22)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m^\omega_{(11)} & m^\omega_{(12)} \\ 0 & m^\omega_{(21)} & m^\omega_{(22)} \end{pmatrix} . \tag{152}$$

Moreover we denote

$$m^\omega_\bullet = \begin{pmatrix} m^\omega_{(11)} & m^\omega_{(12)} \\ m^\omega_{(21)} & m^\omega_{(22)} \end{pmatrix} .$$

The vector function $\eta(x)$ is also decomposed into three parts $\eta_{(0)}(x)$, $\eta_{(1)}(x)$, $\eta_{(2)}(x)$, respectively. We denote for brevity $\eta_\bullet(x) = (\eta_{(1)}(x), \eta_{(2)}(x))$, thus $\eta_\bullet(x)$ is a vector function with values in $\mathbb{R}^{T(n^2+n-1)}$. Similar notations are used for columns as well.

Formulae (102)–(103) and (129) imply that the columns $a(\varepsilon)$ and $b(\varepsilon) - \mathcal{E}\pi^+ v$ are independent of $c_{(0)} = \pi^+_{(0)} v$. Besides that, in view of (128) the equalities $a_1(\varepsilon) = ... = a_T(\varepsilon) = 0$ follow. We recall that $m^\omega_{(00)}, m^\omega_{(01)}, m^\omega_{(02)}$ are null matrices. Taking into account the relations

$$\zeta^j(\xi) = U^j(\xi)\,, \quad \widetilde{\zeta}^j = z^j = 0\,, \quad j = 1, ..., T, \tag{153}$$

we derive the first simplified representation (107) of the approximation $\mathcal{U}$, namely

$$\begin{aligned} \mathcal{U}(\varepsilon, x) = v(x) + \eta_\bullet(x) a_\bullet(\varepsilon) + \widetilde{\zeta}_\bullet(\varepsilon^{-1}x) b_\bullet(\varepsilon) = \\ = v(x) + y_\bullet(x) a_\bullet(\varepsilon) + z_\bullet(x) b_\bullet(\varepsilon)\ . \end{aligned}$$

Inequalities (124) and (125) for the functions u and $\mathcal{U}$ and theirs gradients follow from Proposition 4.1 with $q = 1$. Furthermore, the following estimate is obtained

$$\begin{aligned} |\mathbb{J}_\varepsilon(u) - \mathbb{J}_\varepsilon(\mathcal{U})| \le c_{\mathcal{F}} \int_{\Xi(\varepsilon)} \{|u(\varepsilon,x) - \mathcal{U}(\varepsilon,x)| + |\nabla_x u(\varepsilon,x) - \nabla_x \mathcal{U}(\varepsilon,x)|\} ds_x \le \\ \le c(\mathbf{N}_2) \| u - \mathcal{U}; \Lambda^{2,\alpha}_\beta(\Omega(\varepsilon)) \| \int_{\Xi(\varepsilon)} \{ r^{2+\alpha-\beta} + r^{1+\alpha-\beta} \} ds_x \le C(\mathbf{N}_2) \varepsilon^{n+\delta}\,, \end{aligned}$$

where $c_{\mathcal{F}} = c_{\mathcal{F}}(\max|u|, \max|\mathcal{U}|, \max|\nabla_x u|, \max|\nabla_x \mathcal{U}|)$. Here, we exploit the convergence of integrals over Ξ of functions, $r^{2+\alpha-\beta} = r^{2-n+\delta}$ and $r^{1+\alpha-\beta} = r^{1-n+\delta}$, for $d = n$ and $d = n-1$, and observe that the value of the upper bound in (110) is obtained taking into account (149),

$$c\varepsilon^{n+\delta} \| \{f^\Omega, g^\Omega\}; \mathcal{R}^{1,\alpha}_\beta \Lambda(\Omega, \partial\Omega) \| =: c\varepsilon^{n+\delta} \mathbf{N}_2\ .$$

We are going to apply Lemma 7.1 to the integral $\mathbb{J}_\varepsilon(\mathcal{U})$, and to this end we derive the following inequalities,

$$\begin{aligned} \Big| \int_{\Xi(\varepsilon)} \mathcal{F}(x, v(x) + \eta_\bullet(x) a_\bullet(\varepsilon) + \widetilde{\zeta}_\bullet(\varepsilon^{-1}x) b_\bullet(\varepsilon), \nabla_x \mathcal{U}(\varepsilon,x)) ds_x - \\ - \int_{\Xi(\varepsilon)} \mathcal{F}(x, v(x), \nabla_x \mathcal{U}(\varepsilon,x)) ds_x - \\ - \varepsilon^n \int_{\Xi(\varepsilon)} \mathcal{F}'_u(x, v(x), \nabla_x \mathcal{U}(\varepsilon,x))^\top \eta_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v ds_x \Big| \le \\ \le c(|v|, |\mathcal{U}|, |\nabla_x \mathcal{U}|) \Big\{ \int_{\Xi(\varepsilon)} |\eta_\bullet(x) a_\bullet(\varepsilon) + \widetilde{\zeta}_\bullet(\varepsilon^{-1}x) b_\bullet(\varepsilon)|^2 ds_x + \end{aligned}$$

$$+\int_{\Xi(\varepsilon)} |\eta_{(2)}(x)a_{(2)}(\varepsilon)+\widetilde{\zeta}_{\bullet}(\varepsilon^{-1}x)b_{\bullet}(\varepsilon)|ds_x\Bigg\}\le$$

$$\le c(\mathbf{N}_2)\left\{\int_{\Xi(\varepsilon)}\left|\varepsilon^n r^{1-n}+\varepsilon^{n+1}r^{-n}+\varepsilon\left(\frac{\varepsilon}{r}\right)^{n+1}\right|^2 ds_x+\right.$$

$$\left.+\int_{\Xi(\varepsilon)}\left|\varepsilon^{n+1}r^{-n}+\varepsilon\left(\frac{\varepsilon}{r}\right)^{n+1}\right|ds_x\right\}$$

$$\le c(\mathbf{N}_2)\left\{\varepsilon^{2n}\varepsilon^{2-2n+d}+\varepsilon^{2n+2}\varepsilon^{-2n+d}+\varepsilon^{2n+4}\varepsilon^{-2-2n+d}+\right.$$

$$\left.+\varepsilon^{n+1}\varepsilon^{-n+d}+\varepsilon^{n+2}\varepsilon^{-n-1+d}\right\}\le c(\mathbf{N}_2)\varepsilon^{d+1}$$

In the case of $d=n-1$, the term $\varepsilon^n\int_{\Xi(\varepsilon)}\mathcal{F}'_u(x,v(x),\nabla_x\mathcal{U}(\varepsilon,x))^\top\eta_{(1)}(x)m^{\omega}_{(11)}\pi^+_{(1)}v ds_x$ can be also included in the remainder $O\left(\varepsilon^{d+1}\right)$.
For $d=n$, it follows that

$$\left|\int_{\Xi(\varepsilon)}\mathcal{F}'_u(x,v(x),\nabla_x\mathcal{U}(\varepsilon,x))^\top\eta_{(1)}(x)dx-\int_{\Xi(\varepsilon)}\mathcal{F}'_u(x,v(x),\nabla_x v(x))^\top\eta_{(1)}(x)dx\right|$$

$$\le c(|v|,|\nabla_x v|,|\nabla_x\mathcal{U}|)\int_{\Xi(\varepsilon)}|\nabla_x\eta_{\bullet}(x)a_{\bullet}(\varepsilon)+\varepsilon^{-1}\nabla_\xi\widetilde{\zeta}_{\bullet}(\varepsilon^{-1}x)b_{\bullet}(\varepsilon)||\eta_{(1)}(x)|dx$$

$$\le C(\mathbf{N}_2)\int_{\Xi(\varepsilon)}\left(\varepsilon^n r^{-n}+\varepsilon^{n+1}r^{-n-1}\left(\frac{\varepsilon}{r}\right)^{n+2}\right)r^{1-n}dx\le$$

$$\le C(\mathbf{N}_2)\left(\varepsilon^n\varepsilon^{1-2n+n}+\varepsilon^{n+1}\varepsilon^{-2n+n}+\varepsilon^{n+2}\varepsilon^{-1-2n+n}\right)\le C(\mathbf{N}_2)\varepsilon\,,$$

$$\left|\int_{\omega_\varepsilon}\mathcal{F}'_u(x,v(x),\nabla_x v(x))^\top\eta_{(1)}(x)dx\right|\le c(|v|,|\nabla_x v(x)|)\int_{\omega_\varepsilon}r^{1-n}dx\le c\varepsilon\ .$$

Therefore, in the case $n=d$,

$$\left|\varepsilon^n\int_{\Xi(\varepsilon)}\mathcal{F}'_u(x,v(x),\nabla_x\mathcal{U}(\varepsilon,x))^\top\eta_{(1)}(x)m^{\omega}_{(11)}\pi^+_{(1)}v dx-\right.$$

$$\left.-\varepsilon^n\int_{\Xi}\mathcal{F}'_u(x,v(x),\nabla_x v(x))^\top\eta_{(1)}(x)m^{\omega}_{(11)}\pi^+_{(1)}v dx\right|\le C(\mathbf{N}_2)\varepsilon^{n+1}\,,$$

and for $\Xi=\Omega$, by the general property of the generalised Green's function (see (79)-(80)), it follows that the subtrahend in the left hand-side is equal to

$$\varepsilon^n\int_{\Xi(\varepsilon)}\mathcal{F}'_u(x,v(x),\nabla_x\mathcal{U}(\varepsilon,x))^\top\eta_{(1)}(x)m^{\omega}_{(11)}\pi^+_{(1)}v dx=\varepsilon^n\left(\pi^+_{(1)}V^0\right)^\top m^{\omega}_{(11)}\pi^+_{(1)}v\ ,$$

where V^0 is a solution to the problem of form (147) with a slightly modified right hand side,

$$\mathcal{L}(\nabla_x)V^0(x)+\sum_{p=1}^{\ell}h^p(x)(V^0,h^p)_\Omega=\mathcal{F}'_u(x,v(x),\nabla_x v(x))\,,\quad x\in\Omega\,,\tag{154}$$

$$\mathcal{B}^\Omega(x,\nabla_x)V^0(x)=0\,,\quad x\in\partial\Omega\ .\tag{155}$$

Finally, let us consider the integral, with the notation of Lemma 7.1 in Appendix,

$$\int_{\Xi(\varepsilon)} \mathcal{F}(\underbrace{x, v(x)}_{x}, \underbrace{\nabla_x v(x) + \nabla_x y_\bullet(x) a_\bullet(\varepsilon)}_{\mathfrak{V}(x)} + \underbrace{\varepsilon^{-1}\nabla_\xi z_\bullet(\varepsilon^{-1}x) b_\bullet(\varepsilon)}_{\mathfrak{Y}(\varepsilon^{-1}x) + \mathfrak{Z}(\varepsilon^{-1}x)}) ds_x \; . \tag{156}$$

Since the columns $a_\bullet(\varepsilon)$ and $b_\bullet(\varepsilon)$, included in $\mathfrak{V}, \mathfrak{Y}, \mathfrak{Z}$, depend smoothly on the parameter ε (see Lemma 4.1), in the final formula (222) from Lemma 7.1, we are allowed for further simplifications taking into account the relations

$$\begin{aligned}
\mathfrak{V}(x) &= \nabla_x v(x) + \varepsilon^n \nabla_x y_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v \\
&\quad + \text{small smooth terms of order } O(\varepsilon^{n+\cdots}) \\
&= \nabla_x v(x) + \varepsilon^n \nabla_x y_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v + \ldots \\
\mathfrak{Y}(\xi) &= \nabla_x U^-_{(1)}(\xi)(m^\omega_{(11)} \pi^+_{(1)} v + \text{small columns of order } O(\varepsilon)) \\
&= \nabla_x U^-_{(1)}(\xi) m^\omega_{(11)} \pi^+_{(1)} v + \ldots \\
\mathfrak{Z}(\xi) &= \nabla_\xi \widetilde{\zeta}_{(1)}(\xi)(\pi^+_{(1)} v + \text{small columns of order } O(\varepsilon)) \\
&\quad + \varepsilon \nabla_\xi \widetilde{\zeta}_2 (\text{bounded columns}) = \nabla_\xi \widetilde{\zeta}_{(1)}(\xi) \pi^+_{(1)} v + \ldots
\end{aligned}$$

In the case $d = n-1$ simple considerations based on formula (223) from Lemma 7.1 lead to the result.

Theorem 5.2 *Let $d = n-1$ and $\mathcal{B}^\omega = \mathcal{N}^\omega$, i.e., the Neumann conditions are prescribed on the surface $\partial\omega_\varepsilon$. Consider the solution $u(\varepsilon, x)$ to problem* (3)-(5) *for the particular choice of $g^\omega = 0$ and with $\{f, g\} = \{f^\Omega, g^\Omega + \widehat{g}\}$ which verify formulae* (150), (149) *and* (20), (12). *Then for functional* (2) *the following representation is obtained*

$$\begin{aligned}
&|\mathbb{J}_\varepsilon(u) - \mathbb{J}_0(v) + \varepsilon^{n-1}\mathcal{F}(0, v(0), \nabla_x v(0))\, meas_{n-1}(\omega \cap \Pi) - \\
&-\varepsilon^{n-1} \int_{\Pi \setminus \omega} \{\mathcal{F}(0, v(0), \nabla_x v(0) + \nabla_\xi z_{(1)}(\xi)\pi^+_{(1)} v) - \mathcal{F}(0, v(0), \nabla_x v(0))\} d\xi_\Pi| \leq \\
&\qquad C^\delta_{f,g} \varepsilon^{n-1+\delta} \; ,
\end{aligned} \tag{157}$$

where the constant $C^\delta_{f,g}$ depends on the norm $\|\{f^\Omega, g^\Omega\}; R^{1,\alpha}_\gamma \Lambda(\Omega, \partial\Omega)\|$ and $\delta \in (0, 1/2)$, $v \in \Lambda^{2,\alpha}_\beta(\Omega)^T$ is a solution to problem (17)-(18), *any solution is admissible in the case of nontrivial kernel, $P(\Omega)$ (see Theorem* 2.3*), $\pi^+_{(1)} v$ is the column of derivatives of components v_j at the origin $x = \mathcal{O}$ (see* (151)*), and $z_{(1)} = \left(z^{T+1}, \ldots, z^{T(n+1)}\right)$ the row of energy components $z^j = \zeta^j - U^j$ of the special solutions to the second limit problem defined in Proposition* 3.1. *Integration in* (157) *is performed over the hyperplane Π tangent to the surface Σ.*

Let us consider the case $\Xi = \Omega$, i.e., $d = n$. An application of Lemma 7.1 gives

$$\begin{aligned}
&\mathbb{J}_\varepsilon(u) = \mathbb{J}_0(v) - \varepsilon^n \mathcal{F}(0, v(0), \nabla_x v(0))\, \mathrm{meas}_n \omega + \\
&+\varepsilon^n \int_{\mathbb{R}^n \setminus \omega} \{\mathcal{F}(0, v(0), \nabla_x v(0) + \nabla_\xi z_{(1)}(\xi)\pi^+_{(1)} v) - \mathcal{F}(0, v(0), \nabla_x v(0)) \\
&-\mathcal{F}'_{\nabla u}(0, v(0), \nabla_x v(0))^\top \nabla_\xi U^-_{(1)}(\xi) m^\omega_{(11)} \pi^+_{(1)} v\} d\xi + \\
&+\varepsilon^n (\pi^+_{(1)} V)^\top m^\omega_{(11)} \pi^+_{(1)} v + \varepsilon^n \mathbb{I}(\varepsilon) + o(\varepsilon^{n+\delta}), \quad \delta > 0,
\end{aligned} \tag{158}$$

where we denote

$$\mathbb{I}(\varepsilon) = \int_\Omega \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x))^\top \nabla_x y_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v dx +$$
$$+ \int_{\Omega(\varepsilon)} \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x))^\top \nabla_x U^-_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v dx \ .$$

Remark *In* (158) *and in the sequel the matrix notation is continued to be used. The matrix* $\nabla_\xi U^-_{(1)}$ *of dimension* $n \times Tn$*, here* Tn *is the length of rows* $U^\pm_{(1)}$ *(see* (151)*), contains the rows* $\frac{\partial}{\partial \xi_i} U^-_{(1)}$ *,* $i = 1, ..., n$*. The entries of the rows are simply the* T*-columns of power solutions. The same meaning have the terms* $\nabla_x y_{(1)}$*,* $\nabla_x \eta_{(1)}(x)$*. The* n*-column* $\mathcal{F}'_{\nabla u}$ *consists of* T*-columns*

$$\left(\frac{\partial \mathcal{F}(x, v, \nabla_x u)}{\partial(\partial u_1/\partial x_i)}, ..., \frac{\partial \mathcal{F}(x, v, \nabla_x u)}{\partial(\partial u_T/\partial x_i)} \right)^\top \ , \ i = 1, ..., n \ .$$

Such a notation is consistent with the matrix notation used in the paper and allows for standard by parts integration of terms with the gradient ∇_x*. In particular, the term*

$$\mathcal{F}'_{\nabla u}(x, v(0), \nabla_x v(0))^\top \nabla_\xi U'_{(1)}(\xi)$$

is a row of the length T*, and the expression*

$$\nabla_x^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x))$$

exploited below, denotes a column of the height T*.*

The term $\varepsilon^n \mathbb{I}(\varepsilon)$ can be rewritten in the following way. Integration by parts leads to

$$\mathbb{I}(\varepsilon) = - \int_\Omega \left(\nabla_x^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x)) \right)^\top y_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v dx -$$
$$\underbrace{- \int_{\Omega(\varepsilon)} \left(\nabla_x^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x)) \right)^\top U^-_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v dx +}_{\text{replacing } \Omega(\varepsilon) \text{ with } \Omega}$$
$$+ \int_{\partial\Omega} \left(n(x)^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x)) \right)^\top \underbrace{\left(y_{(1)}(x) + U^-_{(1)}(x) \right)}_{= \eta_{(1)}(x)} m^\omega_{(11)} \pi^+_{(1)} v ds_x$$
$$+ \int_{\partial\omega_\varepsilon} \left(n\left(\frac{x}{\varepsilon}\right)^\top \underbrace{\mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x))}_{\text{freezing at } x=0} \right)^\top U^-_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v ds_x \ ,$$

where replacing $\Omega(\varepsilon)$ with Ω results in an error of the order $O\left(\int_{\omega_\varepsilon} r^{1-n} dx \right) = O(\varepsilon)$, and freezing the coefficients at $x = 0$ in the last line leads to an error of the order $O\left(\int_{\partial\omega_\varepsilon} |x| r^{1-n} ds_x \right) = O(\varepsilon)$.

In this way we obtain

$$\mathbb{I}(\varepsilon) = O(\varepsilon) - \underbrace{\int_\Omega \left(\nabla_x^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x))\right)^\top \eta_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v dx -}_{\mathbf{I}}$$

$$+ \underbrace{\int_{\partial\Omega} \left(n(x)^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x))\right)^\top \eta_{(1)}(x) m^\omega_{(11)} \pi^+_{(1)} v ds_x +}_{\mathbf{II}}$$

$$+ \int_{\partial\omega} \left(n(\xi)^\top \mathcal{F}'_{\nabla u}(0, v(0), \nabla_x v(0))\right)^\top U^-_{(1)}(\xi) m^\omega_{(11)} \pi^+_{(1)} v ds_\xi \ .$$

From basic properties of the generalised Green's functions $\eta_{(1)} = \left(\eta^{1+T}, ..., \eta^{T(n+1)}\right)$, see (79)-(80), the equality follows

$$-\mathbf{I} + \mathbf{II} = (\pi^+ V^1)^\top m^\omega_{(11)} \pi^+_{(1)} v \ ,$$

where V^1 is a solution to the problem

$$\mathcal{L}(\nabla_x) V^1(x) + \sum_{p=1}^{\ell} h^p(x)(V^1, h^p)_\Omega = -\nabla_x^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x)) \, , \quad x \in \Omega \, , \tag{159}$$

$$\mathcal{B}^\Omega(x, \nabla_x) V^1(x) = \mathfrak{J}(x, v(x), \nabla_x v(x)) \ , \tag{160}$$

with the right hand side $\mathfrak{J} = (\mathfrak{J}_1, ..., \mathfrak{J}_T)$ in the boundary condition defined by :

$$\mathfrak{J}_j(x, v(x), \nabla_x v(x)) = \begin{cases} 0 \ , \ \text{if } \mathcal{B}^\Omega_j = S^\Omega_j \text{ see (11)} \ , \\ \left(S^\Omega(x) n(x)^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x))\right)_j \ , \\ \quad \text{if } \mathcal{B}^\Omega_j = \left(\mathcal{S}^\Omega \mathcal{N}^\Omega\right)_j \text{ see (10)} \ . \end{cases} \tag{161}$$

For the Dirichlet or Neumann boundary conditions on $\partial\Omega$ we have

$$\mathfrak{J} = 0 \ , \ \text{or } \mathfrak{J}(x, v(x), \nabla_x v(x)) = n(x)^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x)) \ , \ \text{respectively} \ .$$

In this way we find that the adjoint state is given by the sum $W = V^0 + V^1$, where V^0 and V^1 are solutions to problems (154)-(155) and (159)-(160), respectively.

We complete now the evaluation of the leading terms of asymptotic expansion for functional (2) in the case of $\Xi = \Omega$ and $\mathcal{B}^\omega = \mathcal{N}^\omega$. We want to relax in Theorem 5.3 the presence of nonlocal terms in problem formulation, to this end some properties of the adjoint state W are useful.

Lemma 5.1 *Let $\gamma = \alpha - 2\delta$ and the right hand side* (150) *verify compability conditions* (20), *so that there exists the solution to problem* (17)-(18)

$$v(x) = U_{(0)} \pi^+_{(0)} v + U_{(1)} \pi^+_{(1)} v + U_{(2)} \pi^+_{(2)} v + \widetilde{v}(x) \tag{162}$$

with the remainder $\widetilde{v}(x) \in \Lambda^{2,\alpha}_\gamma(\Omega)^T$. Then the problem

$$\begin{aligned} \mathcal{L}(\nabla_x) W(x) = \mathcal{F}'_u(x, v(x), \nabla_x v(x)) - \nabla_x^\top \mathcal{F}'_{\nabla u}(x, v(x), \nabla_x v(x)) \ , \ x \in \Omega \\ \mathcal{B}^\Omega(x, \nabla_x) W(x) = \mathfrak{J}(x, v(x), \nabla_x v(x)) \ , \ x \in \partial\Omega \ , \end{aligned} \tag{163}$$

where $\mathfrak{J}$ is a column with the components (161), *admit a solution $W \in C^{2,\alpha}(\Omega)^T$, defined up to the elements from the kernel $ker A^\Omega = P(\Omega)$.*

Proof From representation (162) and by the choice of the index γ we inherit that the solution v and its gradient belong to the space $C^{0,\alpha}(\Omega)$. In addition v and ∇v are in $C^{1,\alpha}(\partial\Omega)$, see Section 2.2. It means that the right hand sides in (163) are sufficiently regular to obtain $W \in C^{2,\alpha}(\Omega)^T$ and it remains to check compability conditions (20) which takes the form

$$0 = (\mathcal{F}'_u - \nabla_x^\top \mathcal{F}'_{\nabla u}, p)_\Omega + (\mathfrak{J}, \mathcal{T}^\Omega p)_\Omega = (\mathcal{F}'_u, p)_\Omega + (\mathcal{F}'_{\nabla u}, \nabla_x p)_\Omega \quad \forall p \in P(\Omega) \; , \tag{164}$$

where after integration by parts we take into account the equality $\mathcal{B}^\Omega p = 0$ on $\partial\Omega$. Since functional (2) enjoys the property (89), by the differentiation with respect to t of the identity

$$\int_\Omega \mathcal{F}(x, v(x) + tp(x), \nabla_x v(x) + t\nabla_x p(x))dx = \int_\Omega \mathcal{F}(x, v(x), \nabla_x v(x))dx$$

(164) follows and the proof is complete. □

Now, we pass to the formulation of the obtained result. The accuracy of asymptotic formulae will be $o(\varepsilon^n)$ and therefore, inequality (144) is not sufficient to ignore the correction $\widehat{g}$, however by formulae (150)-(149) in view of $r < c\varepsilon$ for $x \in \omega_\varepsilon$ it follows

$$|(f^\Omega, p)_{\omega_\varepsilon}| \le c\varepsilon^n \varepsilon^{\gamma - n} \|f^\Omega; \Lambda^{0,\alpha}_\gamma(\Omega)\| \le c\varepsilon^{n+2\delta} \|f^\Omega; \Lambda^{0,\alpha}_\gamma(\Omega)\| \; ,$$

which means that, as before, the correction is sufficiently small,

$$\|\widehat{g}; C^{1,\alpha}(\partial\Omega)\| \le c\varepsilon^{n+2\delta} \|f^\Omega; \Lambda^{0,\alpha}_\gamma(\Omega)\| \; . \tag{165}$$

Theorem 5.3 *Let $\Xi = \Omega$ and $\mathcal{B}^\omega = \mathcal{N}^\omega$, i.e., $d = n$ and the Neumann conditions are prescribed on $\partial\omega_\varepsilon$. Let $g^\omega = 0$ and for $\{f, g\} = \{f^\Omega, g^\Omega + \widehat{g}\}$ formulae* (150), (149), (20) *and* (165) *are fulfilled. Then functional* (2)*, evaluated at the solution to problem* (3)-(5) *satisfies the estimate*

$$\begin{aligned} &|\mathbb{J}_\varepsilon(u) - \mathbb{J}_0(v) + \varepsilon^n \mathcal{F}(0, v(0), \nabla_x v(0))\, meas_n \omega - \\ &-\varepsilon^n \int_{\mathbb{R}^n \setminus \omega} \{\mathcal{F}(0, v(0), \nabla_x v(0) + \nabla_\xi z_{(1)}(\xi)\pi^+_{(1)} v) - \mathcal{F}(0, v(0), \nabla_x v(0)) - \\ &\left(\mathcal{F}'_{\nabla u}(0, v(0), \nabla_x v(0))\right)^\top \nabla_\xi U^-_{(1)}(\xi) m^\omega_{(11)} \pi^+_{(1)} v\} d\xi - \\ &-\varepsilon^n \int_{\partial\omega} \left(n(\xi)^\top \mathcal{F}'_{\nabla u}(0, v(0), \nabla_x v(0))\right)^\top U^-_{(1)}(\xi) m^\omega_{(11)} \pi^+_{(1)} v ds_\xi - \\ &-\varepsilon^n (\pi^+_{(1)} W)^\top m^\omega_{(11)} \pi^+_{(1)} v| \le C^\delta_{f,g} \varepsilon^{n+\delta} \; , \end{aligned} \tag{166}$$

where the constant $C^\delta_{f,g}$ depends on the norm $\|\{f^\Omega, g^\Omega\}; R^{1,\alpha}_\gamma(\Omega, \partial\Omega)\|$ and $\delta \in (0, 1/2)$, v is a solution (162) *to problem* (17)-(18)*, any solution is admissible in the case of nontrivial kernel $P(\Omega)$ (see Theorem* 2.3*), $W \in C^{2,\alpha}(\Omega)^T$ is the adjoint state given by a solution to problem* (163)*, finally $m^\omega_{(11)}$ is the part of polarisation matrix defined by* (152) *and $\pi^+_{(1)}$ is the corresponding part of the projection π^+ in* (151).

6 Three dimensional elasticity

We specify the results obtained in the case of elasticity system. Both volume and surface shape functionals are discussed in this section.

6.1 Anisotropic elastic body with a small cavity

Let us consider the elasticity problem written in the matrix/column form

$$\begin{aligned}
&(167) \qquad & \mathcal{L}u = D(-\nabla_x)^\top A D(\nabla_x) u &= 0 \quad \text{in} \quad \Omega(\varepsilon),\\
&(168) \qquad & \mathcal{N}^\Omega u = D(n)^\top A D(\nabla_x) u &= g^\Omega \quad \text{on} \quad \partial\Omega,\\
&(169) \qquad & \mathcal{N}^\omega u = D(n)^\top A D(\nabla_x) u &= 0 \quad \text{on} \quad \partial\omega_\varepsilon,
\end{aligned}$$

where A is a symmetric positive definite matrix of size 6×6 consisting of the elastic material moduli (the Hooke's matrix) and $D(\nabla_x)$ is 6×3-matrix of the first order differential operators,

$$(170) \qquad D(\xi)^\top = \begin{bmatrix} \xi_1 & 0 & 0 & 0 & \alpha\xi_3 & \alpha\xi_2 \\ 0 & \xi_2 & 0 & \alpha\xi_3 & 0 & \alpha\xi_1 \\ 0 & 0 & \xi_3 & \alpha\xi_2 & \alpha\xi_1 & 0 \end{bmatrix} \quad , \quad \alpha = \frac{1}{\sqrt{2}} \; ;$$

u is displacement column, $n = (n_1, n_2, n_3)^\top$ is the unit outward normal vector on $\partial\Omega(\varepsilon)$, i.e. unit column. In this notation the strain and stress columns are given respectively by

$$(171) \qquad D(\nabla_x)u = \epsilon(u) = \left(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \sqrt{2}\epsilon_{23}, \sqrt{2}\epsilon_{31}, \sqrt{2}\epsilon_{12}\right)^\top \; ,$$

$$(172) \qquad AD(\nabla_x)u = \sigma(u) = \left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{31}, \sqrt{2}\sigma_{12}\right)^\top \; .$$

The factors α and $\sqrt{2}$ imply that the norms of strain and stress tensors coincide with the norms of columns (171), (172), respectively. From the latter property in matrix/column notation, any orthogonal transformation of coordinates in $\mathbb{R}^3$ gives rise to the orthogonal transformation of columns (171)-(172) in $\mathbb{R}^6$.

The linear space P in polynomial property (14) becomes the space of rigid motions ,

$$(173) \qquad P = \{d(x)c \; : \; c \in \mathbb{R}^6\} \; , \quad \dim P = 6 \; ,$$

where

$$(174) \qquad d(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & -\alpha x_3 & \alpha x_2 \\ 0 & 1 & 0 & \alpha x_3 & 0 & -\alpha x_1 \\ 0 & 0 & 1 & -\alpha x_2 & \alpha x_1 & 0 \end{bmatrix} .$$

Using the above notation and the Taylor formula we can represent any smooth vector field $\mathfrak{u}$ as follows

$$(175) \qquad \mathfrak{u}(x) = d(x)a + D(x)^\top \epsilon(\mathfrak{u}; \mathcal{O}) + O\left(|x|^2\right) \; ,$$

where $a \in \mathbb{R}^6$ is the rigid motion of $\mathfrak{u}$ at the point $x = \mathcal{O} \in \mathbb{R}^3$ and $\epsilon(\mathfrak{u}; \mathcal{O}) \in \mathbb{R}^6$ is the strain column evaluated at $x = \mathcal{O}$.
The load g^Ω is supposed to be self equilibrated in order to assure the existence of a solution to the elasticity problem,

$$(176) \qquad \int_{\partial\Omega} d(x)^\top g^\Omega(x) ds_x = 0 \in \mathbb{R}^6$$

6.2 Readjustment of the problem

The readjustment of the Neumann problem to uniquely solvable problem of the form (46)-(48) is very useful in many applications and therefore we construct explicitly the functions $\{h^1, ..., h^6\}$ in (43) and (44)-(45), which also makes our presentation of the results in the case of elasticity self-contained.

We denote by $\mathbf{h}$ the matrix $[h^1, ..., h^6]$ with the columns $h^j, j = 1, ..., 6$. We start the construction of $\mathbf{h}$ by introduction of the columns h^j_*, $j = 1, ..., 6$, with $\mathbf{h}_* = [h^1_*, ..., h^6_*]$, where

$$\mathbf{h}_*(x) = \chi_*(r_*) \begin{bmatrix} 1 & 0 & 0 & 0 & -\alpha(x_3 - x_3^*) & \alpha(x_2 - x_2^*) \\ 0 & 1 & 0 & \alpha(x_3 - x_3^*) & 0 & -\alpha(x_1 - x_1^*) \\ 0 & 0 & 1 & -\alpha(x_2 - x_2^*) & \alpha(x_1 - x_1^*) & 0 \end{bmatrix} , \tag{177}$$

and x^* is a point in Ω such that the ball $B^* \subset \mathbb{R}^3$ of the radius R_* with the centre at x^* satisfies $B^* \subset \Omega(\varepsilon)$ for all $\varepsilon > 0$ and sufficiently small. The cutoff function χ_* is supported on B^*, and $r_* = |x - x^*|$.
We find the transformation $\mathbf{d}$ such that $\mathbf{h}(x) = \mathbf{d}^{-1}\mathbf{h}_*(x)$ and $\int_\Omega \mathbf{h}(x)d(x)dx = \mathbb{I}_6$, where $\mathbb{I}_6$ is the identity matrix in $\mathbb{R}^6$, or in another words, $(h^j, p^k)_\Omega = \delta_{j,k}$, while $p^1, ..., p^6$, are columns of $d(x)$.

To this end we evaluate the Gramian d_*,

$$d_* = \int_{B_*} d(x - x_*)^\top d(x - x_*)\chi_*(r_*)dx , \tag{178}$$

d_* is a 6×6 matrix. By symmetry arguments it follows that d_* is a diagonal matrix, $d_* = \mathrm{diag}\{t_1, t_1, t_1, t_2, t_2, t_2\}$, where

$$t_1 = \int_{B_*} \chi_*(r_*)dx = 4\pi \int_0^{R_*} \chi_*(r_*)r_*^2 dr_* > 0 , \tag{179}$$

$$t_2 = \alpha^2 \int_{B_*} \left(|x_i - x_i^*|^2 + |x_j - x_j^*|^2\right) \chi_*(r_*)dx > 0 . \tag{180}$$

Let $c = (c_1', c_2', c_3', c_1'', c_2'', c_3'')^\top$ be a given column in $\mathbb{R}^6$, where $c' = (c_1', c_2', c_3')^\top$, $c'' = (c_1'', c_2'', c_3'')^\top \in \mathbb{R}^3$ and $d(x) = (\mathbb{I}_3, d''(x))$, where $d''(x)$ is a 3×3 matrix. Taking into account the form of $d(x)$ we have

$$d(x)c = d'(x - x^*)c + d''(x^*)c'' . \tag{181}$$

Hence,

$$\int_{B_*} \chi_*(r_*)d(x - x_*)^\top d(x)cdx = d_*c + \int_{B_*} \chi_*(r_*)d(x - x_*)^\top d''(x)c''dx \tag{182}$$

Again, by symmetry arguments the last three elements of the column $\int_{B_*} \chi_*(r_*)d(x-x_*)^\top d''(x)c''dx$ vanish. Thus, the matrix

$$\mathbf{d}_* = \int_{B_*} \chi_*(r_*)d(x - x_*)^\top d(x)cdx \tag{183}$$

is an upper triangular matrix, with the diagonal $\{t_1, t_1, t_1, t_2, t_2, t_2\}$, thus invertible, and enjoys the required property $\mathbf{d}_*^{-1} \int_\Omega \mathbf{h}_*(x)d(x)dx = \mathbb{I}_6$.

6.3 Shape functionals

The general theory presented in the paper can be applied for broad class of shape functionals, however, to fix the ideas we deal only with two representative examples.

Let us consider the functionals

$$\mathbb{J}_\varepsilon^1(u) = \int_{\Omega(\varepsilon)} \sigma(u;x)^\top B(x)\sigma(u;x)dx \ , \tag{184}$$

$$\mathbb{J}_\varepsilon^2(u) = \int_{\Xi(\varepsilon)} b(x')\sigma_{33}(u;x',0)dx' \tag{185}$$

where b is a function on the plane $\Pi = \{x \ : \ x_3 = 0\}$, $x' = (x_1, x_2)^\top$ and $\Xi(\varepsilon) = \Omega(\varepsilon) \cap \Pi$. The functional $\mathbb{J}_\varepsilon^2(u)$ is related to the integral fracture criteria of Neuber-Novozhilov-Weighardt. Functional (184) looks like the elastic energy functional

$$\mathbb{E}(u;\Omega(\varepsilon)) = \frac{1}{2}\int_{\Omega(\varepsilon)} \epsilon(u;x)^\top A\epsilon(u;x)dx = \frac{1}{2}\int_{\Omega(\varepsilon)} \sigma(u;x)^\top A^{-1}\sigma(u;x)dx \tag{186}$$

but can contain a certain symmetric 6×6–matrix function B. In the case of constant, diagonal matrix B functional (184) is related to square of the $L_2(\Omega)$–norm of the stress tensor or of its components. On the other hand, if $A(x)^{-1}B(x)A(x)^{-1}$ becomes a constant diagonal matrix with our choice of B, then in (184) the similar strain norms are obtained.

For problem (167)-(169) the exclusive dependence of the integrand on the displacement vector $u(\varepsilon, x)$ makes no sense, since such a displacement field is defined up to rigid motions (cf. Proposition 2.1 and (173)). On the other hand, if a part Γ_D with $\mathrm{meas}_2(\Gamma_D) > 0$ of the exterior surface $\partial\Omega$ is clamped, (let us note that the clamped interior surface $\partial\omega_\varepsilon$ is not acceptable from the *engineering* point of view), i.e. by the change of condition (168) by the new condition

$$u = 0 \text{ on } \Gamma_D \ , \ \ D(n)^\top AD(\nabla_x)u = 0 \text{ on } \Gamma_N = \partial\Omega \setminus \overline{\Gamma}_D \ ,$$

then functional (13) makes sense for the applications in optimum design. In particular, in the theory of *bracing*, when admissible fixations of the elastic body on the supports is considered, the following functional which describes the mean value of settlement of the body can be introduced

$$\mathbf{J}_\varepsilon(u) = \mathbf{d}^{-1}(\varepsilon)\int_{\Omega(\varepsilon)} d(x)u(\varepsilon,x)dx \ , \ \ \mathbf{d}(\varepsilon) = \int_{\Omega(\varepsilon)} d(x)^\top d(x)dx \tag{187}$$

(compare with (183)). Our results can be applied to scalar functionals defined in terms of (187), but the dependence of such a functional on the cavity is inconsiderable, hence such a functional would not be interesting for applications. Therefore, we restrict ourselves to the particular case of functionals (184), (185) issuing from the fracture mechanics. Let us mention only, in relation to (187), that the dependence of the constant in Korn's inequality on the number and disposition of small parts (of the diameter $O(\varepsilon)$) of the surface $\partial\Omega$ is considered in [51].

6.4 The polarisation matrix (tensor)

We use the columns of $d(x)$ and $D(x)$, namely the basis of polynomials is composed of the functions U^j, $j = 1, ..., 12$,

$$\underbrace{U^1, ..., U^6}_{\text{columns of } d} \ , \quad \underbrace{U^7, ..., U^{12}}_{\text{columns of } D} \ . \tag{188}$$

The functions U^{-k}, $k = 1, ..., 12$, are defined by the formulae

$$U^{-k}(\xi) = \sum_{p=1}^{3} U_p^k(-\nabla_\xi)\Phi^p(\xi) \ , \tag{189}$$

where $\Phi = (\Phi^1, \Phi^2, \Phi^3)$ denotes the fundamental matrix (the Kelvin-Somigliana tensor).

Since $\mathcal{N}^\omega U^j = 0$ on $\partial\omega_\varepsilon$ for the rigid motions $U^1, ..., U^6$, the polarisation 12×12-matrix m^ω defined in (76) contains the first six null columns and the first six null rows. The remaining right lower 6×6-matrix is denoted by $\mathbf{m}^\omega$. In the same way we redenote $\mathbf{U}^{\pm j} = U^{\pm(6+j)}$, $\boldsymbol{\zeta}^j = \zeta^{j+6}$. Thus,

$$\boldsymbol{\zeta}^j(\xi) = \mathbf{U}^j(\xi) + \mathbf{z}^j(\xi) = \mathbf{U}^j(\xi) + \sum_{p=1}^{6} \mathbf{m}_{jp}^\omega \mathbf{U}^{-j}(\xi) + \widetilde{\boldsymbol{\zeta}}^j(\xi) \ , \ \widetilde{\boldsymbol{\zeta}}^j(\xi) = O\left(|\xi|^{-3}\right) \tag{190}$$

(see [72], [55] for the details), and the *energy components* $\mathbf{z}^j$ are given by the solutions to the exterior elasticity problem

$$\mathcal{L}\mathbf{z} = D(-\nabla_\xi)^\top A D(\nabla_\xi)\mathbf{z} = 0 \text{ in } G = \mathbb{R}^3 \setminus \overline{\omega} \ , \tag{191}$$

$$\mathcal{N}^\omega \mathbf{z} = D(n(\xi))^\top A D(\nabla_\xi)\mathbf{z} = \mathbf{g} \text{ on } \partial\omega \tag{192}$$

with the special right hand sides

$$\mathbf{g}^j(\xi) = -D(n(\xi))^\top A D(\nabla_\xi)\mathbf{U}^j(\xi) = -D(n(\xi))^\top A \mathbf{e}^j \ , \tag{193}$$

where $j = 1, ..., 6$ and $\mathbf{e}^j = (\delta_{j,1}, ..., \delta_{j,6})^\top$ is an element of the canonical basis in $\mathbb{R}^6$.

When the tensor notation is used, in contrary to our matrix notation, it turns out that $\mathbf{m}^\omega$ becomes the fourth rank tensor of the same structure and form as the Hooke's tensor. The polarisation matrix is involved in all asymptotic formulae, in particular in the *topological derivatives* obtained in the paper. Therefore, the explicit evaluation of $\mathbf{m}^\omega$ is of importance for practical applications.

For an arbitrary shape of ω this matrix, of course, is not determined explicitly. The same situation occurs even for the fundamental matrix which is not known for an arbitrary anisotropy (see [24]). However, in some particular cases the polarisation matrix can be evaluated.

- For canonical two dimensional and three dimensional bodies, such as a ball, an ellipsoid, ellipsoidal crack, the polarisation matrix is known e.g., in the case of isotropy.
- For two dimensional case the simple formulae obtained by Movchan [41] and reproduced in [42] and the book published by the author are unfortunately not correct. The mistake occurs at the very beginning: formula (6) in [41], has been found by Argatov [2].

Proposition 6.1 *The following integral representations hold true*

$$\mathbf{m}_{jk}^\omega = -\left(D(n)^\top A\mathbf{e}^j, \boldsymbol{\zeta}^k\right)_{\partial\omega} \ , \tag{194}$$

$$\mathbf{m}_{jk}^\omega = \left(AD(\nabla_\xi)\mathbf{z}^j, D(\nabla_\xi)\mathbf{z}^k\right)_G + A_{jk}\mathrm{meas}_3\omega \ , \tag{195}$$

$$\mathbf{m}_{jk}^\omega = \frac{1}{3}\left([n^\top\xi]AD(\nabla_\xi)\boldsymbol{\zeta}^j, D(\nabla_\xi)\boldsymbol{\zeta}^k\right)_{\partial\omega} \ . \tag{196}$$

Proof

Formulae (194) and (195) are derived in [72] (for isotropic bodies), and in [55] (for anisotropic bodies), formula (196) is given in [28], where the question is raised on the equalities between the above formulae. For the convenience of the reader we derive formulae (194) and (195) and show the equivalence with (196), using the known integral representation

$$\mathbf{c}_k = \left(\mathbf{g}, \boldsymbol{\zeta}^k\right)_{\partial\omega} \tag{197}$$

of the coefficients $\mathbf{c}_k$ of $\mathbf{U}^{-k}(\xi)$ in the decomposition at the infinity of the solution $\mathbf{z}$ to problem (191) (cf. calculations (85), (86) in the proof of Proposition 3.2 and see [55]). In view of (190) formula (194) is nothing else but representation (197) in the case of special right hand side (193). Besides,

$$\begin{gathered}\mathbf{m}^{\omega}_{jk} = -\left(\mathbf{g}^j, \mathbf{U}^k + \mathbf{z}^k\right)_{\partial\omega} = \\ \left(D(-n)^\top AD(\nabla_\xi)\mathbf{U}^j, \mathbf{U}^k\right)_{\partial\omega} + \left(D(n)^\top AD(\nabla_\xi)\mathbf{z}^j, \mathbf{z}^k\right)_{\partial\omega} = \\ \left(AD(\nabla_\xi)\mathbf{U}^j, D(\nabla_\xi)\mathbf{U}^k\right)_{\omega} + \left(AD(\nabla_\xi)\mathbf{z}^j, D(\nabla_\xi)\mathbf{z}^k\right)_{\mathbb{R}^3\setminus\omega} .\end{gathered} \tag{198}$$

Here, the Green's formulae are applied for the domains ω and $G = \mathbb{R}^3 \setminus \omega$, taking into account the opposite directions of the normal vectors on $\partial\omega$ for ω and G, respectively. Since $D(\nabla_\xi)\mathbf{U}^j = \mathbf{e}^j$, equalities (198) and (195) coincide.

We turn to identity (196), which needs much more complex verification. The representation of the operator $\mathcal{L}(\nabla_\xi)$ in the spherical coordinates (ρ, φ),

$$\mathcal{L}(\nabla_\xi) = D(-\nabla_\xi)^\top AD(\nabla_\xi) = \rho^{-2}L(\varphi, \nabla_\varphi, \rho\partial_\rho) ,$$

leads to the commuting equality

$$\mathcal{L}(\nabla_\xi)\rho\partial_\rho = (\rho\partial_\rho + 2)\mathcal{L}(\nabla_\xi) , \tag{199}$$

where $\rho\partial_\rho = \rho\dfrac{\partial}{\partial\rho} = \xi^\top\nabla_\xi$.

Let us consider the surface integral

$$I_\Gamma\left(\boldsymbol{\zeta}^j, \boldsymbol{\zeta}^k\right) = \left(D(n)^\top AD(\nabla_\xi)\boldsymbol{\zeta}^j, \rho\partial_\rho\boldsymbol{\zeta}^k\right)_\Gamma - \left(\boldsymbol{\zeta}^j, D(n)^\top AD(\nabla_\xi)\rho\partial_\rho\boldsymbol{\zeta}^k\right)_\Gamma . \tag{200}$$

By (199) the derivative $\rho\partial_\rho\boldsymbol{\zeta}^k$ still satisfies homogeneous system (191), i.e. (200) is an invariant integral independent of the integration surface Γ which encloses the cavity ω.

We now fix $\Gamma = \partial\omega$ and develop (200), employing the boundary condition $\mathcal{N}^\omega\boldsymbol{\zeta}^j = 0$ on $\partial\omega$. Integration by parts, with the cutoff function $\boldsymbol{\chi} \in C_0^\infty(\mathbb{R}^3 \setminus \overline{\omega})$ equal to one in the vicinity of the body $\overline{\omega}$, leads to

$$\begin{aligned} I_{\partial\omega}\left(\boldsymbol{\zeta}^j, \boldsymbol{\zeta}^k\right) &= \left(\mathcal{L}\boldsymbol{\zeta}^j, \rho\partial_\rho\boldsymbol{\chi}\boldsymbol{\zeta}^k\right)_G - \left(\boldsymbol{\zeta}^j, \mathcal{L}\rho\partial_\rho\boldsymbol{\chi}\boldsymbol{\zeta}^k\right)_G = \\ &= \left(\mathcal{L}\boldsymbol{\zeta}^j, \rho\partial_\rho\boldsymbol{\chi}\boldsymbol{\zeta}^k\right)_G - (\boldsymbol{\zeta}^j, \underbrace{\{\rho\partial_\rho + 2\}\mathcal{L}\boldsymbol{\chi}\boldsymbol{\zeta}^k}_{\text{by}(199)})_G = \\ &= \left(\mathcal{L}\boldsymbol{\zeta}^j, \rho\partial_\rho\boldsymbol{\chi}\boldsymbol{\zeta}^k\right)_G + \left(\{\rho\partial_\rho - 1\}\boldsymbol{\zeta}^j, \mathcal{L}\boldsymbol{\chi}\boldsymbol{\zeta}^k\right)_G . \end{aligned}$$

We use here the facts that $\partial_\rho \rho = \rho\partial_\rho + 1$ and furthermore, $\mathcal{L}\chi\boldsymbol{\zeta}^k = 0$ in a neighbourhood of $\partial\omega$. All integrals are convergent due to the presence of the cutoff function χ. To develop further, we take into account the homogeneous boundary conditions on $\partial\omega$ for $\boldsymbol{\zeta}^j$ and $\boldsymbol{\zeta}^k$, so we obtain

$$\begin{aligned} I_{\partial\omega}\left(\boldsymbol{\zeta}^j, \boldsymbol{\zeta}^k\right) &= \left(AD(\nabla_\xi)\boldsymbol{\zeta}^j, D(\nabla_\xi)\rho\partial_\rho\chi\boldsymbol{\zeta}^k\right)_G + \\ &\left(AD(\nabla_\xi)\rho\partial_\rho\boldsymbol{\zeta}^j, D(\nabla_\xi)\chi\boldsymbol{\zeta}^k\right)_G - \left(\boldsymbol{\zeta}^j, \mathcal{L}\chi\boldsymbol{\zeta}^k\right)_G = \\ = \left(AD(\nabla_\xi)\boldsymbol{\zeta}^j, \rho\partial_\rho D(\nabla_\xi)\chi\boldsymbol{\zeta}^k\right)_G &+ \left(A\rho\partial_\rho D(\nabla_\xi)\boldsymbol{\zeta}^j, D(\nabla_\xi)\chi\boldsymbol{\zeta}^k\right)_G + \\ &+2\underbrace{(AD(\nabla_\xi)\boldsymbol{\zeta}^j, D(\nabla_\xi)\chi\boldsymbol{\zeta}^k)_G}_{=0} - \underbrace{(\boldsymbol{\zeta}^j, \mathcal{L}\chi\boldsymbol{\zeta}^k)_G}_{=0} . \end{aligned}$$

We have used the equality $D(\nabla_\xi)\rho\partial_\rho = \{\rho\partial_\rho + 1\}D(\nabla_\xi)$, similar to (199), and we have taken into account that integration by parts of the latter two scalar products result in the linear combination of expressions

$$\left(\mathcal{L}\boldsymbol{\zeta}^j, \chi\boldsymbol{\zeta}^k\right)_G , \ \left(D(n)^\top AD(\nabla_\xi)\boldsymbol{\zeta}^j, \boldsymbol{\zeta}^k\right)_{\partial\omega} , \ \left(\boldsymbol{\zeta}^j, D(n)^\top AD(\nabla_\xi)\boldsymbol{\zeta}^k\right)_{\partial\omega} , \tag{201}$$

which vanish due to the definitions of $\boldsymbol{\zeta}^k$ and $\boldsymbol{\zeta}^j$. So we can complete the transformation

$$\begin{aligned} I_{\partial\omega}\left(\boldsymbol{\zeta}^j, \boldsymbol{\zeta}^k\right) &= \int_G \rho\partial_\rho\{[D(\nabla_\xi)\chi\boldsymbol{\zeta}^k]^\top AD(\nabla_\xi)\boldsymbol{\zeta}^j\}d\xi = \\ +(\xi^\top nAD(\nabla_\xi)\boldsymbol{\zeta}^j, D(\nabla_\xi)\boldsymbol{\zeta}^k)_{\partial\omega} &- 3\underbrace{(AD(\nabla_\xi)\boldsymbol{\zeta}^j, D(\nabla_\xi)\chi\boldsymbol{\zeta}^k)_G}_{=0 \text{ by } (201)} . \end{aligned}$$

It remains to evaluate (200), taking into account asymptotic expansion (190) and by inflating the surface Γ to the infinity. From the equalities

$$\begin{aligned} \rho\partial_\rho\mathbf{U}^j(\xi) = \mathbf{U}^j(\xi) \ , \ \rho\partial_\rho\mathbf{U}^{-k}(\xi) = -2\mathbf{U}^{-k}(\xi) \ , \\ I_\Gamma(\mathbf{U}^j, \mathbf{U}^k) = I_\Gamma(\mathbf{U}^{-j}, \mathbf{U}^{-k}) = 0 \ , \ I_\Gamma(\mathbf{U}^j, \mathbf{U}^{-k}) = 2\delta_{j,k} \ , \ I_\Gamma(\mathbf{U}^{-k}, \mathbf{U}^j) = \delta_{j,k} \end{aligned}$$

(the latter result from the integration by parts either inside Γ, or outside Γ) we obtain

$$I_\Gamma(\boldsymbol{\zeta}^j, \boldsymbol{\zeta}^k) = I_\Gamma(\mathbf{U}^j, \mathbf{m}^\omega_{kj}\mathbf{U}^{-j}) + I_\Gamma(\mathbf{m}^\omega_{kj}\mathbf{U}^{-k}, \mathbf{U}^k) = 3\mathbf{m}^\omega_{kj} .$$

Thus (196) is proved. □

The above result shows that the formulae obtained in [38] and derived in [27] and [28] coincide.

For the numerical methods, the most suitable representation for the polarisation matrix seems to be (194), since it uses only the traces on $\partial\omega$ of solutions $\boldsymbol{\zeta}^k = \mathbf{U}^k + \mathbf{z}^k$. The computations of the decaying in the infinity solutions $\mathbf{z}^k$ to exterior elasticity problems (191)-(192) can be performed by solving the associated integral equations or by an application of the transparent artificial boundary conditions - if the fundamental matrix Φ is known. The new concept of local second order artificial boundary conditions [59] can be used as well, in such a case any knowledge of the fundamental matrix is not at all required and the improved precision given by the appropriate Hölder norm near to the surface $\partial\omega$ is obtained.

6.5 Topological derivatives

From condition (176) follows that both problems, problem (167)-(169) in the body $\Omega(\varepsilon)$ with the cavity ω_ε, and the first limit problem in the entire body Ω,

$$D(-\nabla_x)^\top AD(\nabla_x)v = 0 \text{ in } \Omega \ , \qquad (202)$$
$$D(n)^\top AD(\nabla_x)v = g^\Omega \text{ in } \partial\Omega \ ,$$

admit the solutions $u(\varepsilon, x) \in C^{2,\alpha}(\Omega(\varepsilon))^3$ and $v \in C^{2,\alpha}(\Omega)^3$, respectively, under the loading $g^\Omega \in C^{1,\alpha}(\partial\Omega)^3$. Freedom in selection of such solutions up to the rigid motions from (173) has no influence on functionals (184) and (185) and therefore can be neglected (we recall only, that using the vector functions $h^1, ..., h^6$, constructed in subsection 6.2 we can pass to uniquely solvable problems).

We concretize the asymptotic formulae, derived in Theorems 5.2 and 5.3. We start with the simpler case of functional (185), with the integrand given in (157) of the form

$$\mathcal{F}(0, \epsilon^0(v) + D(\nabla_\xi)\mathbf{z}(\xi)\epsilon^0(v)) - \mathcal{F}(0, \epsilon^0(v)) =$$
$$= \mathcal{F}(0, D(\nabla_\xi)\mathbf{z}(\xi)\epsilon^0(v)) = b(0)\sigma_{33}(\mathbf{z}\epsilon^0(v); \xi) \ .$$

Here and in the sequel we use the following notations :

$$\epsilon^0(v) = \epsilon(v; \mathcal{O}) = (D(\nabla_x)v)(\mathcal{O}) \ , \ \sigma^0(v) = A\epsilon^0(v)$$
$$\sigma_{33}(u) = (\mathbf{e}^3)^\top \sigma(u) = (\mathbf{e}^3)^\top AD(\nabla_x)u$$

(see (171)-(172)); $\mathbf{z} = (\mathbf{z}^1, ..., \mathbf{z}^6)$ is a row of the energy components of the special solutions; we define the rows $\boldsymbol{\zeta}$, $\widetilde{\boldsymbol{\zeta}}$ and $\mathbf{U}^\pm$ in the same way.

Corollary 6.1 *The following formula holds true*

$$\mathbb{J}^2_\varepsilon(u) = \mathbb{J}^2_0(v) - \varepsilon^2 b(0)\{\sigma^0_{33}(v)\text{meas}_2(\Pi \cap \omega) - \int_{\Pi\setminus\omega} \sigma_{33}(\mathbf{z}\epsilon^0(v); \xi', 0)d\xi'\} + O(\varepsilon^{2+\delta}) \ ,$$

where the integration is performed over the plane $\Pi = \{\xi \in \mathbb{R}^3 \ : \ \xi_3 = 0\}$ *and* $\xi' = (\xi_1, \xi_2)$.

Before presenting the result for functional (184), we recall some facts. First of all, the adjoint state $W \in C^{2,\alpha}(\Omega)^3$ is determined from problem (163), i.e.,

$$D(-\nabla_x)^\top AD(\nabla_x)W = -2D(\nabla_x)^\top BAD(\nabla_x)v \text{ in } \Omega \ , \qquad (203)$$
$$D(n)AD(\nabla_x)W = 2D(n)AD(\nabla_x)^\top BAD(\nabla_x)v \text{ on } \partial\Omega \ .$$

Here we observe that functional (184) depends only on $D(\nabla_x)u$, therefore,

$$\mathcal{F}'_{\nabla u}(0, D(\nabla_x)v(0))^\top \nabla_x = \mathcal{F}'_{D(\nabla)u}(0, D(\nabla_x)v(0))^\top D(\nabla_x) \ .$$

Furthermore, we have the relations

$$\mathcal{F}(0, \epsilon^0(v) + D(\nabla_\xi)\mathbf{z}\epsilon^0(v)) - \mathcal{F}(0, \epsilon^0(v)) - \mathcal{F}'_{D(\nabla)u}(0, \epsilon^0(v))^\top D(\nabla_\xi)\mathbf{U}^-(\xi)\mathbf{m}^\omega\epsilon^0(v) =$$
$$= 2\epsilon^0(v)^\top ABAD(\nabla_\xi)\mathbf{z}(\xi)\epsilon^0(v) + (D(\nabla_\xi)\mathbf{z}(\xi)\epsilon^0(v))^\top ABAD(\nabla_\xi)\mathbf{z}(\xi)\epsilon^0(v) -$$
$$-2\epsilon^0(v)^\top ABAD(\nabla_\xi)\mathbf{U}^-(\xi)\mathbf{m}^\omega\epsilon^0(v) =$$
$$= (D(\nabla_\xi)\mathbf{z}(\xi)\epsilon^0(v))^\top ABAD(\nabla_\xi)\mathbf{z}(\xi)\epsilon^0(v) + 2\epsilon^0(v)^\top ABAD(\nabla_\xi)\widetilde{\boldsymbol{\zeta}}(\xi)\epsilon^0(v) \ .$$

Finally, in view of (190),

$$\epsilon^0(v)^\top AB\left\{\int_G AD(\nabla_\xi)\widetilde{\zeta}(\xi)d\xi + \int_{\partial\omega} AD(n)\mathbf{U}^-(\xi)\mathbf{m}^\omega ds_\xi\right\}\epsilon^0(v) =$$
$$= \epsilon^0(v)^\top AB\int_{\partial\omega} AD(n)(\boldsymbol{\zeta}(\xi) - \mathbf{U}^+(\xi))ds_\xi\epsilon^0(v) =$$
$$= -\epsilon^0(v)^\top AB\{\mathbf{m}^\omega + A\mathrm{meas}_3\omega\}\epsilon^0(v)\ .$$

The latter equality follows from (194) and (198). Now, an arrangement of the terms in formula (166) leads to the statement for functional (184).

Corollary 6.2 *The following formula holds true*

$$\begin{aligned}\mathbb{J}^1_\varepsilon(u) = \mathbb{J}^1_0(v) + \varepsilon^3\{&\epsilon^0(v)^\top ABA\epsilon^0(v)\mathrm{meas}_3\omega + \\ &+(ABAD(\nabla_\xi)\mathbf{z}\epsilon^0(v), D(\nabla_\xi)\mathbf{z}\epsilon^0(v))_{\mathbb{R}^3\setminus\omega} + \\ &+(\epsilon^0(W) - 2BA\epsilon^0(v))^\top\mathbf{m}^\omega\epsilon^0(v)\} + O(\varepsilon^{3+\delta})\ ,\end{aligned} \tag{204}$$

where $\epsilon^0(v) = D(\nabla_\xi)v(\mathcal{O})$ and $\epsilon^0(W) = D(\nabla_\xi)W(\mathcal{O})$ are strain columns evaluated at the point $x = \mathcal{O}$ for the solutions of problems (202) *and* (203)*; $\mathbf{m}^\omega$ is the polarisation matrix of size 6×6 for the cavity ω in the elastic space with the Hooke's matrix A, and $\mathbf{z} = (\mathbf{z}^1, ..., \mathbf{z}^6)$ is the row of energy components of the special solutions to homogeneous exterior elasticity problem* (191)*-*(192)*.*

In the particular case of $B(x) = \frac{1}{2}A^{-1}$ functional (184) coincides with the elastic energy (186). Besides, we have $W = v$; thus $\epsilon^0(W) - 2BA\epsilon^0(v) = 0$ and the last term in the parenthesis in (204) vanishes, and by (195) the sum of the first two terms equals $\frac{1}{2}\epsilon^0(v)^\top\mathbf{m}^\omega\epsilon^0(v)$. Thus, we have the relation

$$\mathbb{E}(u;\Omega(\varepsilon)) = \mathbb{E}(u;\Omega) + \frac{1}{2}\varepsilon^3\epsilon^0(v)^\top\mathbf{m}^\omega\epsilon^0(v) + O(\varepsilon^{3+\delta})\ . \tag{205}$$

As it is well known, the potential energy of the strain/stress state

$$\mathbb{P}(u;\Omega(\varepsilon)) = \mathbb{E}(u;\Omega(\varepsilon)) - \int_{\partial\Omega} u(\varepsilon,x)^\top g^\Omega(x)ds_x\ , \tag{206}$$

evaluated on the solutions of problem (167)-(169), coincides with $-\mathbb{E}(u;\Omega(\varepsilon))$, so that formula (205) transforms to the formula

$$\mathbb{P}(u;\Omega(\varepsilon)) = \mathbb{P}(u;\Omega) - \frac{1}{2}\varepsilon^3\epsilon^0(v)^\top\mathbf{m}^\omega\epsilon^0(v) + O(\varepsilon^{3+\delta}) \tag{207}$$

and becomes of the same form as the formulae given in [33], [71], [72], [48], [55], [28], [69]. Theorems 5.1, 135, 136 covers as well some other results on asymptotic analysis of functionals, in particular the paper [13].

The assumption $n \geq 3$ is used only to simplify the presentation of the results derived in the paper. Since in the case of $n = 2$ the fundamental matrix $\Phi(x)$ contains $\log|x|$, the factor $\log\varepsilon$ appears in formulae (102), (103), however the asymptotic analysis performed in Section 4 is still significant. We point out that it is already shown in [14], [16], [35], [33], ([38], Chapter 2, 4) that in the case of two dimensional Dirichlet problem (3)-(4) the coefficients (102) and (103) becomes fractional functions of $\log \varepsilon$, but the remainders retain the power order of decaying with $\varepsilon \to 0+$ (we note that in [13] the error is only $O(|\log \varepsilon|^{-2})$). The relation between such asymptotic forms and the Padé approximation is discussed in [3].

We consider only the operator $\mathcal{L}(\nabla_x)$ with the constant coefficients, however the main results of the paper (in particular Theorems 5.1, 5.2, 5.3 and Corollaries 6.1, 6.2) remain the same for the operators with variable coefficients. However, dealing with such operators necessitates more complex notation and requires more refined and sophisticated matching procedures. We refer the reader to monographs [38]-[39] and paper [20] for such investigations.

6.6 Modelling of small defects in elastic bodies

In the paper [13] an approach is proposed to replace the problem in variable domain $\Omega(\varepsilon)$ with $\varepsilon \in (0,1]$ by a problem in the fixed geometrical domain but with the operator depending on the small parameter ε. This approach is introduced in order to apply the standard Lagrangian formalism in shape optimisation, we refer e.g. to [10] for the description of the formalism. To this end, an artificial truncation boundary $\mathbb{S}_R = \partial \mathbb{B}_R$ is introduced around ω_ε with a fixed radius R, and the boundary condition of integral type is imposed on the sphere $\mathbb{S}_R$,

$$\mathcal{N}^{\mathbb{S}_R}(x,\nabla_x)u(\varepsilon,x) := D\left(-\frac{x}{R}\right)^\top AD(\nabla_x)u(\varepsilon,x) = \left(T^{\varepsilon,R}u\right)(\varepsilon,x)\ ,\ x \in \mathbb{S}_R\ .$$

Here $T^{\varepsilon,R}$ stands for Steklov-Poincaré operator at the exterior surface of the elastic body $\mathbb{B}_R \setminus \omega_\varepsilon$, i.e., the mapping

$$H^{1/2}(\mathbb{S}_R)^3 \ni w \mapsto T^{\varepsilon,R}w := \mathcal{N}^{\mathbb{S}_R}W \in H^{-1/2}(\mathbb{S}_R)^3\ , \tag{209}$$

where $W \in H^1(\mathbb{B}_R \setminus \omega_\varepsilon)^3$ denotes the solution of the problem

$$\mathcal{L}W = 0 \ \text{ in } \mathbb{B}_R \setminus \overline{\omega_\varepsilon}\ ,$$
$$\mathcal{N}^\omega W = 0 \text{ on } \partial\omega_\varepsilon\ ,\ W = w \text{ on } \mathbb{S}_R\ .$$

The operator $T^{\varepsilon,R}$ absorbs all dependence of problem (167)-(168) on the parameter ε and on the form of the cavity ω and enjoys worthy properties which allow to reformulate the problem exclusively in the domain $\Omega \setminus \mathbb{B}_R$. Although determination of the influence of ε and ω on operator (209) itself needs an asymptotic analysis on the same level of complexity as we performed in sections 3 and 4, this approach mimics the classical matching procedure and is rather intrinsic for the method of matched asymptotic expansions (cf. [12]). We emphasise that the polarisation matrix $\mathbf{m}^\omega$, known explicitly for the isotropic elasticity with the spherical cavity $\omega_\varepsilon = \mathbb{B}_\varepsilon$, implicitly emerges in work [13].

We outline another mathematical approach [53], [50], [55] of modelling of various small defects, invoked from the theory of selfadjoint extensions and issuing from the method [5], [60] in diffraction theory. The same kind of modelling of defects in elastic bodies is performed formally in [21]. For simplicity and brevity of presentation we limit ourselves, in the same framework as in [13], to the considerations concerning functional (13), independent of the gradients ∇u, with the Dirichlet condition fulfilled on the whole surface $\partial\Omega(\varepsilon)$,

$$\mathcal{L}(\nabla_x)u = f \text{ in } \Omega(\varepsilon)\ ,\ u = 0 \text{ on } \partial\Omega(\varepsilon) \tag{210}$$

which is simple in the sense that we could consider only the first terms of the asymptotic expansions for the solution. We refer the reader to [50] for the asymptotic approximations of arbitrary order and, in particular, for the Neumann boundary conditions (169).

Let us consider the first limit problem

$$\mathcal{L}(\nabla_x)v = f \text{ in } \Omega\ ,\ v = 0 \text{ on } \partial\Omega \tag{211}$$

and associate with the problem the unbounded operator $\mathfrak{A}$ in the Hilbert space $L_2(\Omega)^3$, with the given differential expression $\mathcal{L}(\nabla_x)$ and the domain of definition

$$\mathcal{D}(\mathfrak{A}) = \{\mathfrak{v} \in H^2(\Omega)^3\ :\ \mathfrak{v} = 0 \text{ on } \partial\Omega\ ,\ \mathfrak{v}(\mathcal{O}) = 0\}\ . \tag{212}$$

The condition $\mathfrak{v}(\mathcal{O}) = 0$ in (212) imitates the boundary condition $u = 0$ on $\partial\omega_\varepsilon$. In view of the embedding $H^2(\Omega) \subset C(\Omega)$ the operator $\mathfrak{A}$ is closed, since the expression $\mathcal{L}(\nabla_x)$ is formally selfadjoint, the operator $\mathfrak{A}$ is symmetric.

The following result can be found, e.g. in [50].

Lemma 6.1 **1)** *Adjoint operator $\mathfrak{A}^*$ is defined by the same differential expression $\mathcal{L}(\nabla_x)$, with the domain of definition*

$$\begin{aligned}\mathfrak{D} = \mathcal{D}(\mathfrak{A}^*) = \{v \in L_2(\Omega)\ :\ v(x) = v^0(x) + \chi(x)[c + \Phi(x)a]\ ,\\ v^0 \in \mathcal{D}(\mathfrak{A})\ ,\ c, a \in \mathbb{R}^3\}\ ,\end{aligned} \tag{213}$$

where $\Phi = (\Phi^1, \Phi^2, \Phi^3)$ is the fundamental matrix for the operator $\mathcal{L}(\nabla_x)$ in $\mathbb{R}^3$.

2) *The restriction $\mathfrak{A}^\omega_\varepsilon$ of the operator $\mathfrak{A}^*$ to the linear space*

$$\mathcal{D}(\mathfrak{A}^\omega_\varepsilon) = \{v \in \mathcal{D}(\mathfrak{A}^*)\ :\ a = \varepsilon m^\omega_{(0)} c\} \tag{214}$$

is a selfadjoint extension of $\mathfrak{A}$. In (214) $m^\omega_{(0)}$ stands for the main (upper left) 3×3-block of the polarisation matrix m^ω, i.e., $\left(-m^\omega_{(0)}\right)$ is the elastic capacity matrix for the cavity ω in the elastic space with the Hooke's matrix A.

The solution $v^\varepsilon \in \mathcal{D}(\mathfrak{A}^\omega_\varepsilon)$ of the abstract equation

$$\mathfrak{A}^\omega_\varepsilon v^\varepsilon = f \in L_2(\Omega)^3 \tag{215}$$

takes the form

$$v^\varepsilon(x) = v(x) + \eta_{(0)}(x)a_{(0)}(\varepsilon)\ , \tag{216}$$

where $\eta_{(0)} = (\eta^1, \eta^2, \eta^3)$ is the Green's matrix for problem (211) with the poles at the point $x = \mathcal{O}$ (columns η^k are solutions to problem (79)-(80)). From the relations introduced in (214) it follows

$$\begin{aligned}a = a_{(0)}(\varepsilon) \in \mathbb{R}^3\ ,\ c = v(\mathcal{O}) + m^\Omega_{(0)}a_{(0)}(\varepsilon)\ \Rightarrow\\ a_{(0)}(\varepsilon) = \varepsilon m^\omega_{(0)}(v(\mathcal{O}) + m^\Omega_{(0)}a_{(0)}(\varepsilon))\ \Rightarrow\\ a_{(0)}(\varepsilon) = \{\mathcal{I} - \varepsilon m^\omega_{(0)}m^\Omega_{(0)}\}^{-1}\varepsilon m^\omega_{(0)}v(\mathcal{O})\ .\end{aligned} \tag{217}$$

If we compare (102), (91), taking into account notation (151), it is easy to see that expression (216) does coincide with the expansion in (91) for the solution to problem (211).

In [50] it is shown that the energy functional

$$\mathfrak{E}^\omega_\varepsilon(v^\varepsilon, f) = \frac{1}{2}(\mathfrak{A}^\omega_\varepsilon v^\varepsilon, v^\varepsilon)_\Omega - (f, v^\varepsilon)_\Omega\ ,$$

evaluated at the solution of equation (215) approximates potential energy functional (206) for problem (211) with the precision $O(\varepsilon^{1+\delta})$, $\delta \in (0, 1/2)$. Beside that, in [20] (see also [48]) it is proved that for any $t > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the eigenvalues $\lambda_n(\varepsilon) \in (0, t)$ of the operator $\mathfrak{A}^\omega_\varepsilon$ are related to the eigenvalues $\Lambda_n(\varepsilon)$ of the spectral Dirichlet problem for the operator $\mathcal{L}(\nabla_x)$ in $\Omega(\varepsilon)$:

$$|\Lambda_n(\varepsilon) - \lambda_n(\varepsilon)| \leq c_n^\delta \varepsilon^{1+\delta} .$$

In other words, the operator $\mathfrak{A}^\omega_\varepsilon$ with the abstract equation (215) asymptotically acquires the attributes of problem (211). The following simple statement confirms that the same holds true for a class of the shape functionals.

Lemma 6.2 *Assume that for* $\varepsilon = 0$ *the functional* $\mathbb{J}_0(u) = \int_\Omega \mathcal{F}(x, u(x)) ds_x$, *defined in* (13) *with* $\Xi(\varepsilon) = \Omega(\varepsilon)$, *is continuous on the space* $L_p(\Omega)^T$, $p \in [1, 3)$, *and for all* $v \in H^2(\Omega)^T$, $w \in L_p(\Omega)^T$

$$\left| \mathbb{J}_0(v + w) - \mathbb{J}_0(v) - \int_\Omega \mathcal{F}'_u(x, v(x))^\top w(x) dx \right| \leq C(v) \|w; L_p(\Omega)^T\|^{1+\delta} \ , \ \delta > 0 \tag{218}$$

(such conditions are fulfilled, e.g., for a quadratic functional). Then, for the solution u *of problem* (211) *and the solution* v^ε *of abstract equation* (215) *the inequality is valid*

$$|\mathbb{J}_0(v^\varepsilon) - \mathbb{J}_\varepsilon(u(\varepsilon, \cdot))| \leq C\varepsilon^{1+\delta} .$$

Proof
Since the Green's matrix $\eta_{(0)}(x) = O(|x|^{-1})$ belongs to the space $L_p(\Omega)^{T \times T}$ with $p \in [1, 3)$, by the condition (218) we have

$$\left| \mathbb{J}_0(v^\varepsilon) - \mathbb{J}_0(v) - \int_\Omega \mathcal{F}'_u(x, v(x))^\top \eta_{(0)}(x) a_0(\varepsilon) dx \right| \leq C |a_0(\varepsilon)|^{1+\delta} .$$

On the other hand, from (217)

$$|a_0(\varepsilon) - \varepsilon m^\omega_{(0)} v(\mathcal{O})| \leq C\varepsilon^2 \ , \ |a_0(\varepsilon)| \leq C\varepsilon \ ,$$

thus by Theorem 19(1)

$$|\mathbb{J}_0(v^\varepsilon) - \underbrace{\{\mathbb{J}_0(v) + \varepsilon \int_\Omega \mathcal{F}'_u(x, v(x))^\top \eta_{(0)}(x) dx \ m^\omega_{(0)} v(\mathcal{O})\}}_{= \mathbb{J}_\varepsilon(u) + O(\varepsilon^{1+\delta}) \text{ by } (145)}| \leq C\varepsilon^{1+\delta} .$$

□

Unfortunately, the domain of definition (214) still depends on ε. In order to avoid such dependence, in [50] is proposed the equivalent formulation of equation (215) in the space $\mathfrak{D}$ with detached asymptotics (see (213)).

To this end, we reduce the first limit problem (211) in the punctured domain $\Omega \setminus \mathcal{O}$, enlarge the solution space to Hilbert space (213) with the norm

$$\|v; \mathfrak{D}\| = (\|v^0; H^2(\Omega)\|^2 + |a|^2 + |c|^2)^{1/2} \ ,$$

and impose *the point condition* which compensates for this enlargement and involves the projection $\pi^{\pm}_{(0)}$ (see (151)) as well as reflects the relation indicated in (213). As a result we obtain the boundary value problem

$$\mathcal{L}(\nabla_x)v^{\varepsilon}(x) = f(x) \ , \ x \in \Omega \setminus \mathcal{O} \ , \tag{219}$$

$$v^{\varepsilon}(x) = 0 \ , \ x \in \partial\Omega \ , \tag{220}$$

$$\pi^{-}_{(0)}v^{\varepsilon} = \varepsilon m^{\omega}_{(0)}\pi^{+}_{(0)}v^{\varepsilon} \ . \tag{221}$$

which is well posed with the unique solution, and this solution takes form (216). Furthermore, problem (219)-(221) can be reformulated as a variational problem, i.e., it is equivalent to the problem of minimisation of a quadratic functional. We refer the reader to [50] for details. The same kind of construction can be devised for higher order approximations of boundary value problems in $\Omega(\varepsilon)$, in particular to Neumann problem (167)-(169). The useful comments on the issue can be found in [55]. Finally we refer to [46] for applications to problems with unilateral conditions and to [44] for the applications to crack problems. Such modelling in the fixed geometrical domain setting can be applied in shape optimisation in particular in the framework of the Lagrangian formalism and therefore to replace *ad hoc* chosen formulations with the Steklov-Poincaré operators.

7 Appendix

The following lemma describes the first term of asymptotic expansion of the integral functional (133) (cf. (156)).

Lemma 7.1 *Assume that* $\mathfrak{F} \in C^2(\overline{G} \times \mathbb{C}^T)$, $\mathfrak{V} \in C^1(\overline{G})^T$, $\mathfrak{Y}$, $\mathfrak{Z} \in C^1(\mathbb{R}^n \setminus \omega)$, *the function* $\mathfrak{Y}$ *is positively homogeneous of degree* $(-n)$, *i.e.,*

$$\mathfrak{Y}(\xi) = |\xi|^{-n}\mathfrak{Y}(|\xi|^{-1}\xi) \ ,$$

and

$$|\mathfrak{Z}(\xi)| \le c|\xi|^{-n-1} , \quad |\nabla_{\xi}\mathfrak{Z}(\xi)| \le c|\xi|^{-n-2} \ .$$

Then the following inequalities are valid.

$$\begin{aligned} &\Bigg| \int_{\Xi(\varepsilon)} \mathfrak{F}\left(x, \mathfrak{V}(x) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) ds_x \\ &- \int_{\Xi} \mathfrak{F}\left(x, \mathfrak{V}(x)\right) ds_x + \varepsilon^d \mathfrak{F}(0, \mathfrak{V}(0)) meas_d(\omega) - \varepsilon^n \int_{\Xi(\varepsilon)} \mathfrak{F}'_u(x, \mathfrak{V}(x))^{\top}\mathfrak{Y}(x)\, ds_x \\ &-\varepsilon^n \int_{\mathbb{R}^n\setminus\omega} [\mathfrak{F}(0, \mathfrak{V}(0) + \mathfrak{Y}(\xi) + \mathfrak{Z}(\xi)) - \mathfrak{F}(0, \mathfrak{V}(0)) - \mathfrak{F}'_u(0, \mathfrak{V}(0))^{\top}\mathfrak{Y}(\xi)]\, d\xi \Bigg| \le \\ &\le c(\mathfrak{F}, \mathfrak{V}, \mathfrak{Y}, \mathfrak{Z}, \Xi, \omega)\varepsilon^{d+\delta} \ , \end{aligned} \tag{222}$$

where $\delta > 0$ *is small.*

2) *If* $d = n - 1$, *and* Π *denotes the tangent hyperplane to* Ξ *at the origin* $\mathcal{O}$, ξ_{Π} *is the projection of* ξ *onto* Π, *and in inequality* (222) *we replace* $\mathbb{R}^n \setminus \omega$ *by* $\Pi \setminus \omega$, $d\xi$ *by* $d\xi_{\Pi}$, *and*

$meas_d(\omega)$ by $meas_d(\Pi\cap\omega)$, respectively, the following inequality holds,

$$\begin{aligned}(223)\qquad &\left|\int_{\Xi(\varepsilon)} \mathfrak{F}\left(x, \mathfrak{V}(x)+\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) ds_x - \right.\\ &-\int_{\Xi} \mathfrak{F}\left(x, \mathfrak{V}(x)\right) ds_x + \varepsilon^d \mathfrak{F}(0,\mathfrak{V}(0)) meas_d(\Pi\cap\omega)\\ &\left. -\varepsilon^d \int_{\Pi\setminus\omega} \left[\mathfrak{F}(0,\mathfrak{V}(0)+\mathfrak{Y}(\xi)+\mathfrak{Z}(\xi)) - \mathfrak{F}(0,\mathfrak{V}(0))\right] d\xi_{\Pi}\right| \leq\\ &\leq c(\mathfrak{F},\mathfrak{V},\mathfrak{Y},\mathfrak{Z},\Xi,\omega)\varepsilon^{d+\delta}\,,\end{aligned}$$

where $\delta>0$ is small.

Proof

Let $\mathbb{B}=\{x\ :\ |x|<\varepsilon^{\mu}\}$, where $\mu\in(0,1)$ is a number to be selected.

First, we consider the case of $d=n$. Using the decomposition $\Xi(\varepsilon)=(\Xi\setminus\mathbb{B})\cup(\Xi(\varepsilon)\cap\mathbb{B})$, we start with the estimates for integrals on $\Xi\setminus\mathbb{B}$ and on $\Xi(\varepsilon)\cap\mathbb{B}$. We denote for simplicity

$$\mathfrak{H}_1(x)=\mathfrak{F}\left(x,\mathfrak{V}(x)+\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right)-\mathfrak{F}(x,\mathfrak{V}(x))-\mathfrak{F}'_u(x,\mathfrak{V}(x))^{\top}\left(\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right)\,.$$

From the second order Taylor expansion, since the derivative $\mathfrak{F}''_u$ is bounded, it follows that

$$(224)\qquad \left|\int_{\Xi\setminus\mathbb{B}}\mathfrak{H}_1(x)dx\right|\leq C\int_{\Xi\setminus\mathbb{B}}\left|\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right|^2 dx\leq C\varepsilon^{2n}\varepsilon^{\mu(n-2n)}$$

and

$$\begin{aligned}(225)\qquad &\left|\int_{\Xi\setminus\mathbb{B}}\left\{\mathfrak{F}\left(x,\mathfrak{V}(x)+\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right)-\mathfrak{F}(x,\mathfrak{V}(x))\right\}dx\right.\\ &\left.-\int_{\Xi\setminus\mathbb{B}}\mathfrak{F}'_u(x,\mathfrak{V}(x))^{\top}\left(\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right)dx\right|\\ &\leq C\int_{\Xi\setminus\mathbb{B}}\left|\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right|^2 dx\leq C\varepsilon^{2n}\varepsilon^{\mu(n-2n)}\,.\end{aligned}$$

Here, the first order term can be estimated

$$(226)\qquad \left|\int_{\Xi\setminus\mathbb{B}}\mathfrak{F}'_u(x,\mathfrak{V}(x))^{\top}\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)dx\right|\leq C\varepsilon^{n+1}\varepsilon^{\mu(n-n-1)}\,,\quad \textit{where we need } \mu<1\,.$$

Now, it remains to estimate the integrals on $\Xi(\varepsilon)\cap\mathbb{B}$, we denote again by

$$\mathfrak{H}_2(x)=\mathfrak{F}\left(x,\mathfrak{V}(x)+\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right)-\mathfrak{F}\left(0,\mathfrak{V}(0)+\mathfrak{Y}\left(\frac{x}{\varepsilon}\right)+\mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right)$$

and apply the Taylor formula to get

$$(227)\qquad \left|\int_{\Xi(\varepsilon)\cap\mathbb{B}}\mathfrak{H}_2(x)dx\right|\leq c\int_{\mathbb{B}}|x|dx\leq c\varepsilon^{(n+1)\mu}\,,\quad \textit{where we require } \mu>\frac{n}{n+1}\,,$$

furthermore

$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{F}\left(x, \mathfrak{V}(x) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) dx \right.$$

$$\left. - \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) dx \right|$$

$$\leq c \int_{\mathbb{B}} |x| dx \leq c\varepsilon^{(n+1)\mu}, \quad \textit{where we require } \mu > \frac{n}{n+1}$$

and

$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{F}\left(0, \mathfrak{V}(0)\right) dx + \varepsilon^n \mathfrak{F}\left(0, \mathfrak{V}(0)\right) \operatorname{meas}_n \omega - \int_{\mathbb{B}} \mathfrak{F}\left(x, \mathfrak{V}(x)\right) dx \right| \leq \tag{228}$$

$$\leq c \int_{\mathbb{B}} |x| dx \leq c\varepsilon^{(n+1)\mu}, \quad \textit{where we require } \mu > \frac{n}{n+1} \ ,$$

finally

$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \left[\mathfrak{F}'_u(0, \mathfrak{V}(0))^\top \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) - \mathfrak{F}'_u(x, \mathfrak{V}(x))^\top \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) \right] dx \right| \leq \tag{229}$$

$$\leq c \int_{\mathbb{B}} |x| \left(\frac{r}{\varepsilon}\right)^{-n} dx \leq c\varepsilon^{(n+1)\mu}, \quad \textit{where we require } \mu > \frac{n}{n+1} \ .$$

Again, we denote for $d = n$,

$$\mathfrak{H}_3(x) = \mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) - \mathfrak{F}\left(0, \mathfrak{V}(0)\right) - \mathfrak{F}'_u(0, \mathfrak{V}(0))^\top \mathfrak{Y}\left(\frac{x}{\varepsilon}\right)$$

$$-\varepsilon^d \left\{ \mathfrak{F}(0, \mathfrak{V}(0) + \mathfrak{Y}(\xi) + \mathfrak{Z}(\xi)) - \mathfrak{F}(0, \mathfrak{V}(0)) - \mathfrak{F}'_u(0, \mathfrak{V}(0))^\top \mathfrak{Y}(\xi) \right\}$$

and we have

$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{H}_3(x) dx \right| \leq c \left(\varepsilon^{n+1} \varepsilon^{-\mu} + \varepsilon^{2n} \varepsilon^{-\mu n} \right) , \tag{230}$$

$$\textit{here any } \mu \in (0,1) \textit{ is admissible} \ .$$

We have also

$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) dx \right.$$

$$- \int_{\Xi(\varepsilon)\cap\mathbb{B}} \left\{ \mathfrak{F}\left(0, \mathfrak{V}(0)\right) + \mathfrak{F}'_u(0, \mathfrak{V}(0))^\top \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) \right\} dx -$$

$$\left. -\varepsilon^d \int_{\mathbb{R}^n\setminus\omega} \left\{ \mathfrak{F}(0, \mathfrak{V}(0) + \mathfrak{Y}(\xi) + \mathfrak{Z}(\xi)) - \mathfrak{F}(0, \mathfrak{V}(0)) - \mathfrak{F}'_u(0, \mathfrak{V}(0))^\top \mathfrak{Y}(\xi) \right\} d\xi \right| \leq$$

$$\leq c \int_{\mathbb{R}^n\setminus\mathbb{B}} \left(\left| \mathfrak{Z}\left(\frac{x}{\varepsilon}\right) \right| + \left| \mathfrak{Z}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) \right|^2 \right) dx \leq$$

$$\leq c \int_{\mathbb{R}^n\setminus\mathbb{B}} \left(\varepsilon^{n+1} r^{-n-1} + \varepsilon^{2n} r^{-2n} \right) dx \leq c \left(\varepsilon^{n+1} \varepsilon^{-\mu} + \varepsilon^{2n} \varepsilon^{-\mu n} \right) ,$$

$$\textit{here any } \mu \in (0,1) \textit{ is admissible} \ .$$

For $\mu = \dfrac{n+1}{n+2}$, using the relation $\mathfrak{Y}(\frac{x}{\varepsilon}) = \varepsilon^n \mathfrak{Y}(x)$, it follows that left hand side of (222) can be bounded from above using (224)–(230).

In the second case $d = n-1$ we repeat (224)–(230) with some simplifications. We start with

(231)
$$\left| \int_{\Xi \setminus \mathbb{B}} \left(\mathfrak{F}\left(x, \mathfrak{V}(x) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) - \mathfrak{F}(x, \mathfrak{V}(x)) \right) ds_x \right.$$
$$\leq c \int_{\Xi \setminus \mathbb{B}} \left| \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right|^2 ds_x \leq C\varepsilon^{2n}\varepsilon^{\mu(d-2n)}\,, \quad \textit{where we need } \mu < 1\,,$$

furthermore

(232)
$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{F}\left(x, \mathfrak{V}(x) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) ds_x \right.$$
$$\left. - \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) ds_x \right|$$
$$\leq c \int_{\mathbb{B}\cap\Xi} |x| ds_x \leq c\varepsilon^{(d+1)\mu}\,, \quad \textit{where we require } \mu > \frac{d}{d+1}$$

and

(233)
$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \left(\mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) - \mathfrak{F}(0, \mathfrak{V}(0)) \right) ds_x - \right.$$
$$\left. \int_{(\Pi\setminus\omega_\varepsilon)\cap\mathbb{B}} \left(\mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) - \mathfrak{F}(0, \mathfrak{V}(0)) \right) dx_\Pi \right| \leq$$
$$\leq c \int_{\Xi(\varepsilon)\cap\mathbb{B}} \left(|x| + |x|^2 \left(\left|\nabla_x \mathfrak{Y}\left(\frac{x}{\varepsilon}\right)\right| + \left|\nabla_x \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right| \right)\right) ds_x \leq$$
$$\leq c \int_{\Xi(\varepsilon)\cap\mathbb{B}} \left(r + r^2 \left(\varepsilon^n \frac{1}{r^{n+1}} + \varepsilon^{n+1} \frac{1}{r^{n+2}} \right)\right) ds_x \leq$$
$$\leq c \left(\varepsilon^{(d+1)\mu} + \varepsilon^n \varepsilon^{(d+1-n)\mu} + \varepsilon^{n+1}\varepsilon^{(d-n)\mu} \right)\,, \text{ here we require } \mu > \frac{d}{d+1}\,.$$

Finally we have

$$\left| \int_{\Xi(\varepsilon)\cap\mathbb{B}} \mathfrak{F}\left(0, \mathfrak{V}(0)\right) ds_x + \varepsilon^d \mathfrak{F}\left(0, \mathfrak{V}(0)\right) \operatorname{meas}_d\left(\Pi \cap \omega\right) - \int_{\mathbb{B}} \mathfrak{F}\left(x, \mathfrak{V}(x)\right) ds_x \right| \leq$$
$$\leq c \int_{\mathbb{B}\cap\Xi} |x| dx \leq c\varepsilon^{(d+1)\mu}$$

and

(234)
$$\left| \int_{(\Pi\setminus\omega_\varepsilon)\cap\mathbb{B}} \left(\mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right) - \mathfrak{F}\left(0, \mathfrak{V}(0)\right) \right) dx_\Pi \right.$$
$$\left. -\varepsilon^d \int_{\Pi\setminus\omega} \left(\mathfrak{F}\left(0, \mathfrak{V}(0) + \mathfrak{Y}\left(\xi\right) + \mathfrak{Z}\left(\xi\right)\right) - \mathfrak{F}\left(0, \mathfrak{V}(0)\right)\right) d\xi_\Pi \right| \leq$$
$$\leq c \int_{\Pi\setminus\mathbb{B}} \left| \mathfrak{Y}\left(\frac{x}{\varepsilon}\right) + \mathfrak{Z}\left(\frac{x}{\varepsilon}\right)\right| dx_\Pi \leq c \int_{\Pi\setminus\mathbb{B}} \frac{\varepsilon^n}{r^n} d\xi_\Pi \leq c\varepsilon^n\,.$$

Combining the estimates (231)–(234), the inequality (222) follows. Let us point out that in comparison with the case $d = n$, the following terms can be neglected in (222) for $d = n - 1$,

$$-\varepsilon^n \int_{\Xi(\varepsilon)} \mathfrak{F}'_u(x, \mathfrak{V}(x))^\top \mathfrak{Y}(x)\, ds_x \ ,$$

$$-\varepsilon^d \int_{\Pi\setminus\omega} \mathfrak{F}'_u(0, \mathfrak{V}(0))^\top \mathfrak{Y}(\xi)\, d\xi_\Pi \ .$$

Remark *In both cases of the dimension d the integral over $\Pi \setminus \omega$ in (222) is convergent, where for $d = n$ we have $\Pi \setminus \omega = \mathbb{R}^n \setminus \omega$. For $d = n$, in general the integral*

$$\varepsilon^d \int_{\Pi\setminus\omega} \mathfrak{F}(0, \mathfrak{V}(0) + \mathfrak{Y}(\xi) + \mathfrak{Z}(\xi))\, d\xi_\Pi$$

is divergent. The precise analysis of the convergence of this integral allows us to determine the number of terms depending on the derivatives which should be subtracted from the integrand $\mathfrak{F}(0, \mathfrak{V}(0) + \mathfrak{Y}(\xi) + \mathfrak{Z}(\xi))$ in order to assure the convergence of the resulting integral. The subtracted terms are rewritten in the slow variable ξ, the integral over $\Xi(\varepsilon)$ in the second line of (222) can serve as an example. The number of derivatives depends on the rate of decay of the function $\mathfrak{Y}(\xi)$ at infinity and on the dimensions d and n. If, for example $\mathfrak{Y}(\xi) = \varepsilon^{1-n}\mathfrak{Y}(x)$, then for $d = n = 3$ we should replace the integrand $f(0, \mathfrak{V}(0) + \mathfrak{Y}(\xi) + \mathfrak{Z}(\xi))$ by the integrand in the form of the difference

$$\begin{aligned} &\mathfrak{F}(0, \mathfrak{V}(0) + \mathfrak{Y}(\xi) + \mathfrak{Z}(\xi)) \\ -\mathfrak{F}(0, \mathfrak{V}(0)) - \mathfrak{F}'_u(0, \mathfrak{V}(0))^\top(\mathfrak{Y}(\xi) + \mathfrak{Z}(\xi)) &- \mathfrak{Y}(\xi)^\top \mathfrak{F}''_{uu}(0, \mathfrak{V}(0))\mathfrak{Y}(\xi) \end{aligned} \tag{235}$$

in order to assure the convergence of the integral over $\Pi \setminus \omega$.

References

[1] G. Allaire, E. Bonnetier, G. Francfort, F. Jouve *Shape optimization by the homogenization method* Numer. Math. **76**(1997) 27-68.

[2] I. I. Argatov *Integral characteristics of rigid inclusions and cavities in the two-dimensional theory of elasticity* Prikl. Mat. Mekh. **62**(1998) 283–289. (Engl. transl. in J. Appl. Maths Mechs, **62**(1998) 263–268)

[3] I. I. Argatov *Refinement of the asymptotic solution obtained by the method of matched expansions in contact problem of elasticity theory* Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, **40**(2000) 623-632. (Engl. transl. in Computational Mathematics and Mathematical Physics, **40**(2000) 594-603)

[4] M.Ph. Bendsoe *Optimization of Structural Topology, Shape and Material* Springer, Berlin, 1995.

[5] F.A. Berezin, L.D.Faddeev *Remark on the Schrödinger equation with singular potential* Dokl Akad Naul SSSR **137**(1961) 1011-1014. (Engl transl. in Soviet Math. Dokl **2**(1961) 372-375)

[6] W. Borchers, K. I. Pileckas *Existence,uniqueness and asymptotics of steady jets* Arch. Rat. Mech. Anal. **120**(1992) 1-49.

[7] A. Campbell, S. A. Nazarov *Une justification de la méthode de raccordement des développements asymptotiques appliquée a un probléme de plaque en flexion. Estimation de la matrice d'impedance* J. Math. Pures Appl. **76**(1997) 15-54.

[8] A. Campbell, S. A. Nazarov *Comportement d'une plaque élastique dont une petite région est rigid et animé d'un mouvement vibratoire. Étude asymptotique de la matrice d'impedance* Ann. Fac. Sci. Toulouse Math. Sér. 6. **4**(2)(1995) 211-242.

[9] A. V. Cherkaev, Y. Grabovsky, A. B. Movchan, S. K. Serkov *The cavity of the optimal shape under the shear stresses* Int. J. Solids *&* Structures, **25**(1999) 4391-4410.

[10] M.C. Delfour, J.-P. Zolésio *Shapes and Geometries: Analysis, Differential Calculus, and Optimization* SIAM series on Advances in Design and Control, Philadelphia, 2001.

[11] H.A. Eschenauer, V.V. Kobelev, A. Schumacher *Bubble method for topology and shape optimization of structures* Struct. Optimiz. **8**(1994) 42-51.

[12] R.R Gadyl'shin *Existence and asymptotics of poles with a small imaginary part for the Helmholtz resonator* (Russian) Uspekhi Mat. Nauk **52**(1)(1997) 3-76. (Engl transl. in Russian Math. Surveys **52**(1)(1997) 1–72)

[13] S. Garreau, Ph. Guillaume, M. Masmoudi, *The topological asymptotic for PDE systems: the elasticity case* SIAM Journal on Control and Optimization. **39**(6)(2001) 1756–1778.

[14] A. M. Il'in, *A boundary value problem for the elliptic equation of second order in a domain with a narrow slit. I. The two-dimensional case.* Mat.sb. **99**(141)(1976) 514-537. (Engl transl. in Math USSR Sbornik **28**(1976))

[15] A. M. Il'in, *Study of the asymptotic behavior of the solution of an elliptic boundary value problem in a domain with a small hole* Trudy Sem. Petrovsk. **6**(1981) 57-82 (Russian)

[16] A. M. Il'in, *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, Translations of Mathematical Monographs, Vol. 102, AMS 1992.

[17] L. Jackowska-Strumiłło, J. Sokołowski, A. Żochowski *The topological derivative method and artificial neural networks for numerical solution of shape inverse problems*, RR-3739, INRIA-Lorraine, 1999.

[18] L. Jackowska-Strumiłło, J. Sokołowski, A. Żochowski, A. Henrot *On numerical solution of shape inverse problems*, *Computational Optimization and Applications* **23**(2002), 231-255.

[19] M. Kachanov, I. Tsukrov, B. Shafiro, *Effective moduli of solids with cavities of various shapes* Appl. Mech. Rev. **47**(1)(1994) S151-S174.

[20] I. V. Kamotski, S. A. Nazarov *Spectral problems in singular perturbed domains and selfadjoint extensions of differential operators* Trudy St.-Petersburg Mat. Obshch. **6**(1998) 151-212. (Engl. transl. in Proceedings of the St. Petersburg Mathematical Society, **6**(2000) 127-181, Amer. Math. Soc. Transl. Ser. 2, 199, Amer. Math. Soc., Providence, RI)

[21] S.K. Kanaun, V.M. Levin *The effective field method in the mechanics of composite materials* (Russian) Izdatel'stvo Petrozavodskogo Universiteta, Petrozavodsk, 1993.

[22] V. A. Kondrat'ev *Boundary problems for elliptic equations in domains with conical or angular points* Trudy Moskov. Mat.Obshch. **16**(1967) 209-292. (Engl. transl. in Trans. Moscow Math. Soc. **16**(1967) 227-313)

[23] V. A. Kozlov, V. G. Maz'ya, J. Rossmann *Elliptic boundary value problems in domains with point singularities* Amer. Math. Soc, Providence, 1997.

[24] V. Kupradze, T. Gegelia, M. Basheleishvili, T. Burchuladze *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity* North Holland, Amsterdam, 1979.

[25] D. Leguillon, E. Sanches-Palencia *Computation of singular solutions in elliptic problems and elasticity* Paris, Masson. 1987

[26] T. Lewinski, J. Sokolowski *Optimal shells formed on a sphere. The topological derivative method.* RR-3495, INRIA-Lorraine, 1998.

[27] T. Lewinski, J. Sokolowski *Topological derivative for nucleation of non-circular voids* Contemporary Mathematics, American Math. Soc. **268**(2000), 341-361.

[28] T. Lewinski, J. Sokolowski *Energy change due to appearing of cavities in elastic solids* Les prépublications de l'Institut Élie Cartan 23/2001, to appear in *Int. J. Solids & Structures.*

[29] T. Lewinski, J. Sokolowski, A. Żochowski *Justification of the bubble method for the compliance minimization problems of plates and spherical shells* CD-Rom, 3rd World Congress of Structural and Multidisciplinary Optimization (WCSMO-3) Buffalo/Niagara Falls, New York, May 17-21, 1999

[30] J.L. Lions, E. Magenes *Problèmes aux limités non homogènes* 1968, Dunod, Paris.

[31] V. G. Maz'ya, B. A. Plamenevskii *Estimates in L_p and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary* Math. Nachr. **81**(1978) 25-82. (Engl. transl. in Amer. Math. Soc. Transl. (Ser.2) **123**(1984) 1-56)

[32] V. G. Maz'ya, N. F. Morozov, B. A. Plamenevskii *On nonlinear bending of a plate with a crack* Differential and integral equations. Boundary value problems (I.N. Vekua Memorial collection) Tbilisi, 1979, 145-163. (Engl. transl. in Amer. Math. Soc. Transl. (Ser.2) **123**(1984) 125-139)

[33] V. G. Maz'ya, S. A. Nazarov *The asymptotic behavior of energy integrals under small perturbations of the boundary near corner points and conical points* Trudy Moskov. Mat. Obshch. **50**(1987) 79-129. (English transl. in Trans. Mosc. Math. Soc. **50**(1988) 77-127)

[34] V. G. Maz'ya, S. A. Nazarov, B. A. Plamenevskii *Asymptotic behavior of solutions of a quasi-linear equation in nonregular perturbed domains* Differentsial'nye Uravnenia i ih Primenen., Vilnius, Lith. Acad. Sci. **27**(1980) 17-50. (Russian)

[35] V. G. Maz'ya, S. A. Nazarov, B. A. Plamenevskii *On the asymptotic behavior of solutions of elliptic boundary value problems with irregular perturbations of the domain* Probl. Mat. Anal., Leningrad, Leningrad Univ. **8**(1981) 72-153. (Russian)

[36] V. G. MAZ'YA, S. A. NAZAROV, B. A. PLAMENEVSKII *Asymptotic expansions of the eigenvalues of boundary value problems for the Laplace operator in domains with small holes* Izv. Akad. Nauk SSSR. Ser. Mat. **48**(2)(1984) 347-371. (English transl. in Math. USSR Izvestiya. **24**(1985) 321-345)

[37] V. G. MAZ'YA, S. A. NAZAROV, B. A. PLAMENEVSKII *Asymptotics of solutions to elliptic boundary-value problems under a singular perturbation of the domain* Tbilisi: Tbilisi Univ. 1981 (Russian)

[38] W. G. MAZJA, S. A. NAZAROV, B. A. PLAMENEVSKII *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. 1.* Berlin: Akademie-Verlag. 1991. (English transl.: Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. 1, Basel: Birkhäuser Verlag, 2000)

[39] W. G. MAZJA, S. A. NAZAROV, B. A. PLAMENEVSKII *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. 2.* Berlin: Akademie-Verlag. 1991. (English transl.: Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. 2, Basel: Birkhäuser Verlag, 2000)

[40] W. G. MAZJA, S. A. NAZAROV, B. A. PLAMENEVSKII *On the singularities of solutions of the Dirichlet problem in the exterior of a slender cone* Mat. sbornik. **122**(4)(1983) 435-456. (English transl. in Math. USSR Sbornik. **50**(1985) 415-437)

[41] A.B. MOVCHAN *Wiener polarization and capacity matrices for the operator of the theory of elasticity in doubly connected regions* Mat. zametki, **47**(2)(1990) 151-153.

[42] A.B. MOVCHAN *Integral characteristics of elastic inclusions and cavities in the two-dimensional theory of elasticity* Eur. J. Appl.Math. **3**(1992) 21-30.

[43] S. A. NAZAROV, B. A. PLAMENEVSKY, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, De Gruyter Exposition in Mathematics 13, Walter de Gruyter, 1994.

[44] S. A. NAZAROV, O.R. POLYAKOVA *Rupture criteria, asymptotic conditions at crack tips, and selfadjoint extensions of the Lamé operator* Trudy Moskov. Mat. Obshch., **57**(1996) 16–75. (English transl. in Trans. Moscow Math. Soc. **57**(1996) 13–66.

[45] S. A. NAZAROV *Self-adjoint boundary value problems. The polynomial property and formal positive operators* Probl. Mat. Anal., St.-Petersburg: St.-Petersburg Univ. **16**(1997) 167-192. (Russian)

[46] S. A. NAZAROV *Asymptotic solution to a problem with small obstacles* Differentsial'nye Uravneniya. **31**(6)(1995) 1031-1041. (English transl. in Differential equations. **31**(6)(1995) 965-974)

[47] S. A. NAZAROV *Asymptotic expansions of eigenvalues* Leningrad Univ. 1987 (Russian)

[48] S. A. NAZAROV *Two-term asymptotics of solutions of spectral problems with singular perturbations* Mat. sbornik. **69**(3)(1991) 291-320. (English transl. in Math. USSR. Sbornik. **69**(2)(1991) 307-340)

[49] S. A. NAZAROV, YU. A. ROMASHEV *Variation of the intensity factor under rupture of the ligament between two collinear cracks* Izv. Akad. Nauk Armenian SSR. Mekh. **4**(1982) 30-40 (Russian)

[50] S. A. NAZAROV *Asymptotic conditions at points, selfadjoint extensions of operators and the method of matched asymptotic expansions* Trudy St.-Petersburg Mat. Obshch. **5**(1996) 112-183 (Russian) (English transl. in Trans. Am. Math. Soc. Ser. 2. **193**(1999) 77-126)

[51] S. A. NAZAROV *Korn's inequalities for junctions of spatial bodies and thin rods* Math. Methods Appl. Sci., **20**(3)(1997) 219–243.

[52] S. A. NAZAROV *The polynomial property of self-adjoint elliptic boundary-value problems and the algebraic description of their attributes* Uspekhi mat. nauk. **54**(5)(1999) 77-142 (Russian) (English transl. in Russian Math. Surveys **54**(5)(1999) 947-1014)

[53] S. A. NAZAROV *Selfadjoint extensions of the Dirichlet problem operator in weighted function spaces* Mat. sbornik. **137**(2)(1988) 224-241. (English transl. in Math. USSR Sbornik. **65**(1)(1990) 229-247)

[54] S. A. NAZAROV *Concentrated masses problems for a spatial elastic body* C. R. Acad. Sci. Paris. Sér. 1. **316**(12)(1993) 1329-1334.

[55] S. A. NAZAROV *The damage tensor and measures. 1. Asymptotic analysis of anisotropic media with defects* Mekhanika tverd. tela. 3(2000) 113-124. (Russian) (English transl. in Mechanics of Solids bf 35(3)(2000) 96-105)

[56] S. A. NAZAROV, M. SPECOVIUS-NEUGEBAUER *Approximation of unbounded domains by bounded domains. Boundary-value problems for the Lamé operator* Algebra Analiz. **8**(5)(1996) 229-268. (English transl. in St. Petersburg Math. J. **8**(5)(1997) 879-912)

[57] S. A. NAZAROV, M. SPECOVIUS-NEUGEBAUER *Selfadjoint extensions of the Neumann Laplacian in domains with cylindrical outlets* Comm. in Math. Physics. **185**(3)(1997) 689-707.

[58] S. A. NAZAROV, M. SPECOVIUS-NEUGEBAUER *The errors due to approximating unbounded elastic bodies by bounded ones* Prikl. Mat. Mekh. **62**(4)(1998) 650-663. (English transl. in J. Appl. Math. Mech. **62**(4)(1998) 605-616)

[59] S. A. NAZAROV, M. SPECOVIUS-NEUGEBAUER *Artificial boundary conditions for elliptic systems in domains with conical outlets to infinity* Dokl. Ross. Akad. Nauk. **377**(3)(2001) 1-4. (Russian)

[60] B.S.PAVLOV *The theory of extension and explicitly soluble models* Uspehi Mat. Nauk **42**(6)(1987) 99-131. (Engl transl. in Soviet Math. Surveys **42**(6)(1987) 127-168)

[61] G. PÓLYA, G. SZEGÖ *Isoperimetric inequalities in mathematical phisics* Princeton University Press, Princeton, NJ, 1951.

[62] M. SCHIFFER, G. SZEGÖ *Virtual mass and polarization.* Trans. Amer. Math. Soc. **67**(1949) 130-205.

[63] SHIN OSAVA *Singular Hadamard's variation of domains and eigenvalues of Laplacian, Parts 1,2* Proc. Japan Acad Sci **56**(1980) 306-310, **57**(1981) 242-246.

[64] V.I. SMIRNOV, *A Course of Higher Mathematics* Pergamon Press, New York, 1964.

[65] J. SOKOŁOWSKI, J-P. ZOLESIO, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer Verlag, 1992.

[66] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *On topological derivative in shape optimization*, SIAM Journal on Control and Optimization. **37**(4)(1999) 1251-1272.

[67] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *Topological derivative for optimal control problems*, Control and Cybernetics **28**(3)(1999) 611-626.

[68] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *Topological derivatives for elliptic problems*, Inverse Problems, **15**(1)(1999) 123-134.

[69] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *Topological derivatives of shape functionals for elasticity systems* Mechanics of Structures and Machines 29(3)(2001) 333-351.

[70] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *Optimality conditions for simultaneous topology and shape optimization* Les prépublications de l'Institut Élie Cartan 8/2001, submitted to *SIAM Journal on Control and Optimization.*

[71] I. S. ZORIN, A. B. MOVCHAN, S. A. NAZAROV *Application of the elastic polarization tensor in the problems of the crack mechanics* Mekhanika tverd. tela. (6)(1988) 128-134. (Russian)

[72] I. S. ZORIN, A. B. MOVCHAN, S. A. NAZAROV *Application of tensors of elastic capacity, polarization and associated deformation* Studies in elasticity and plasticity. Leningrad, Leningrad Univ. (16)(1990) 75-91. (Russian)

[73] M.D. VAN DYKE *Perturbation methods in fluid mechanics* Academic Press, New York. 1964

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